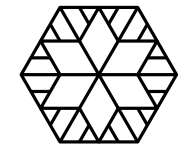
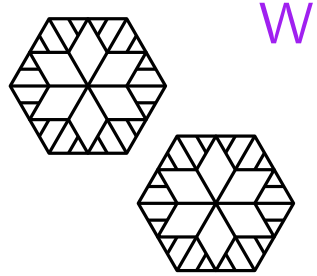
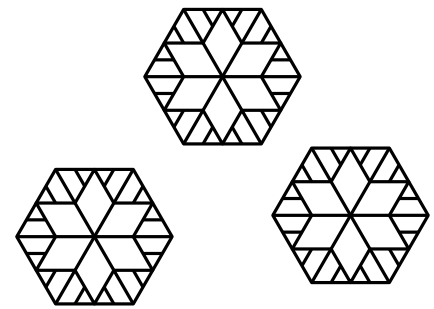
Introduction to **Graph Minor theory** and the **Irrelevant Vertex technique**



Laure Morelle

Winter School





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- if you don't understand something
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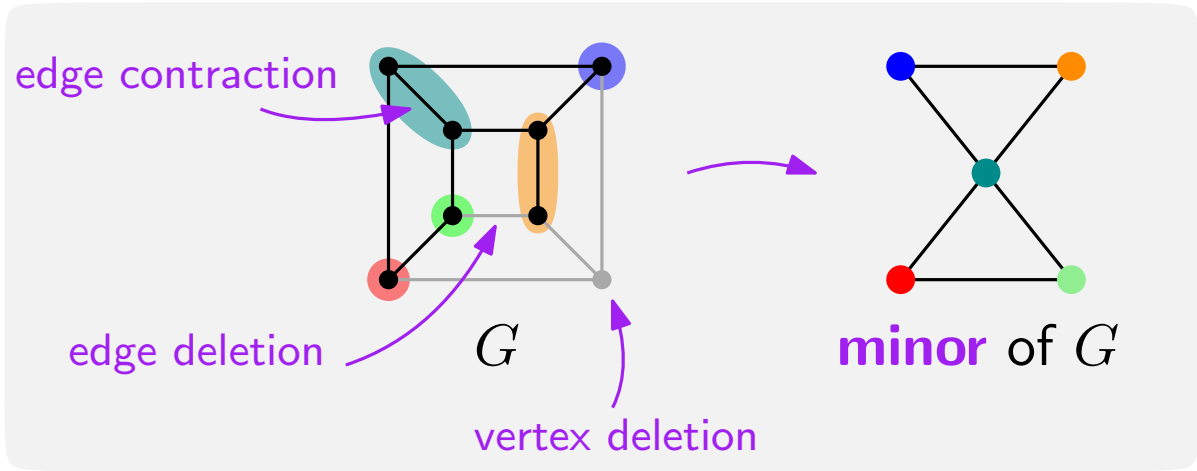
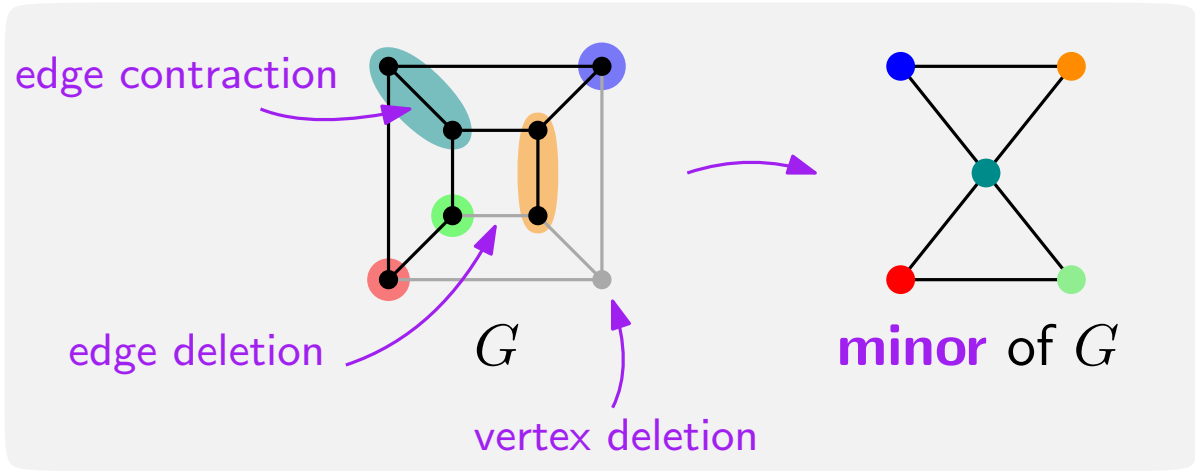
Also:

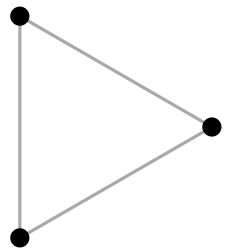
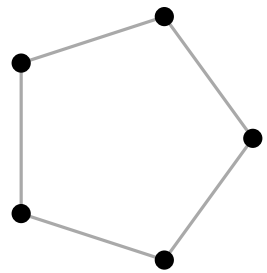
Take a paper and a pen please.

Minors


edge contraction
edge deletion
G
vertex deletion
minor of G

Minors



Is  a minor of  ?

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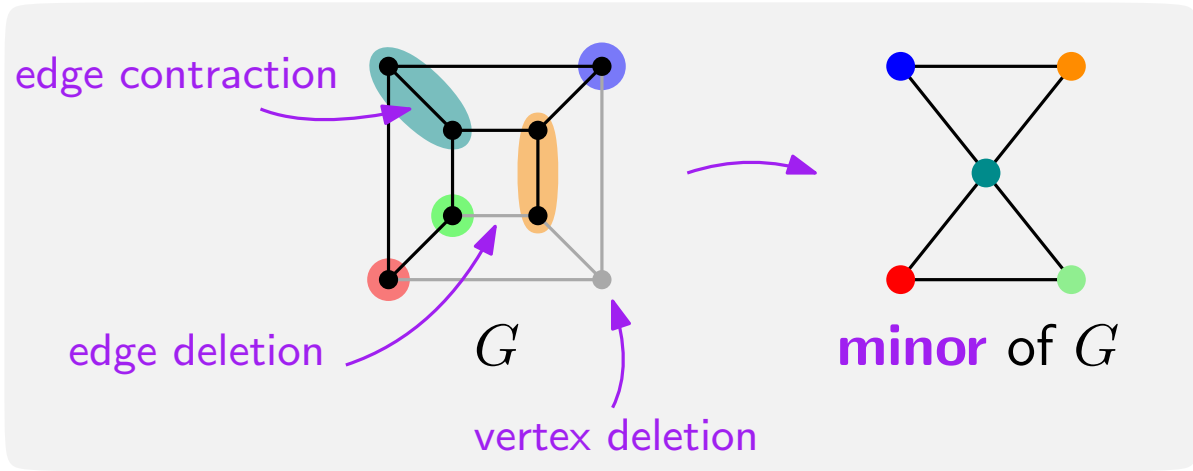
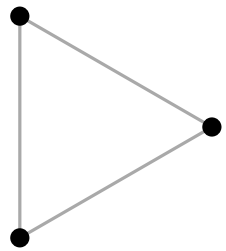
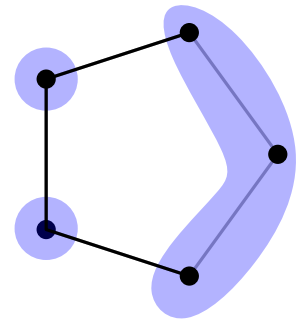
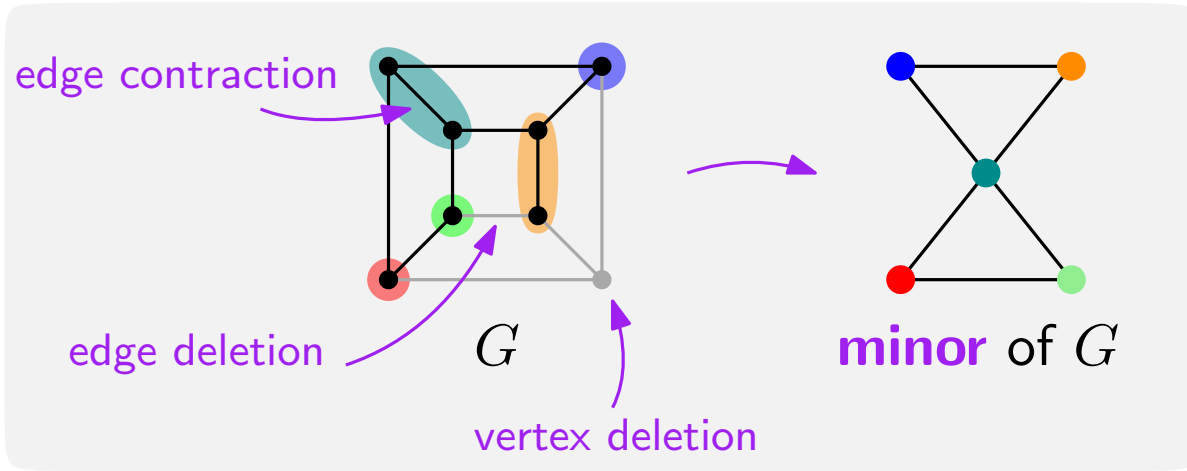
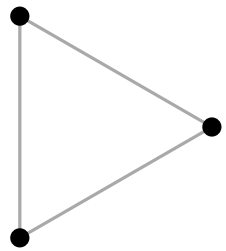
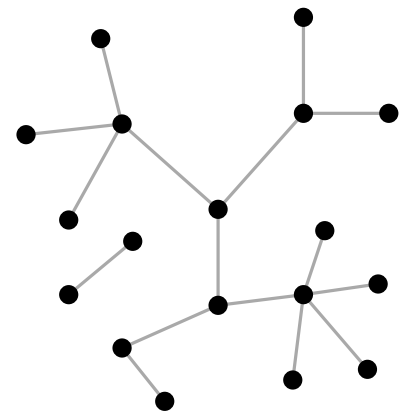


Is  a minor of  ? yes

K_3 is a minor of every cycle.

Minors



Is  a minor of  ?

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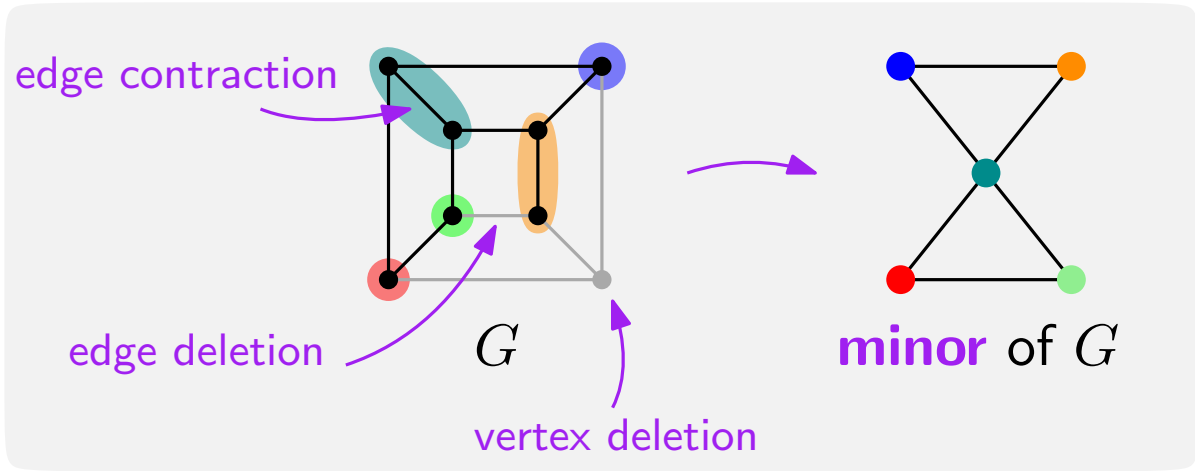
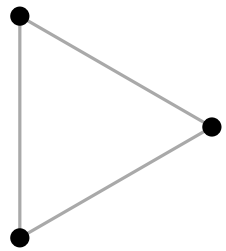
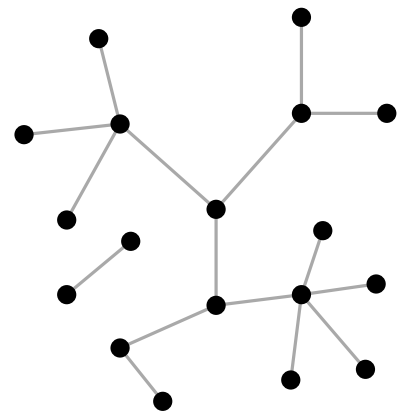
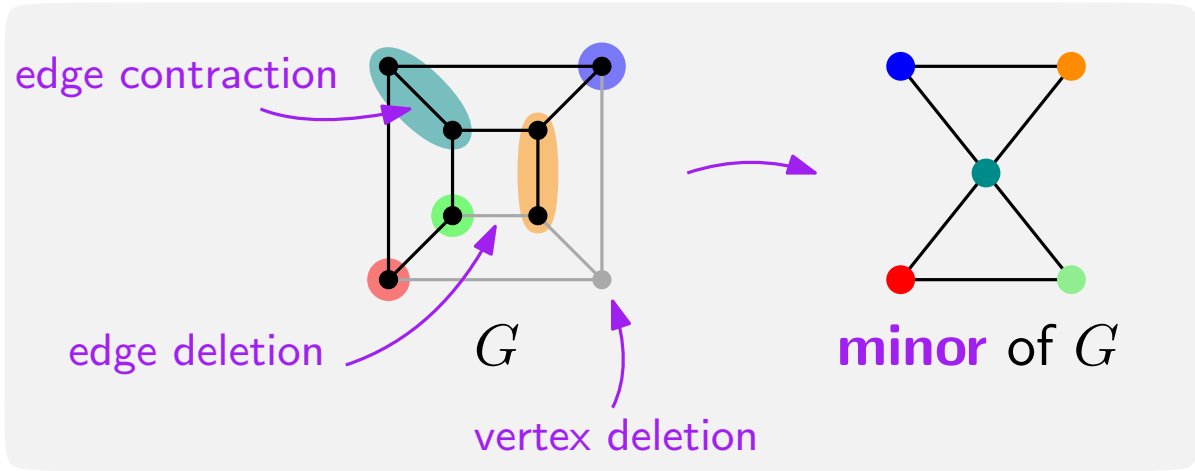
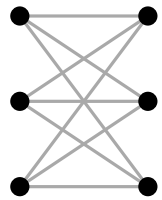
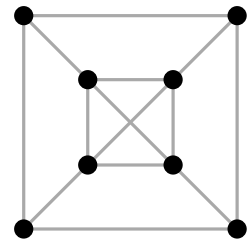


Is  a minor of  ? No

forests $=$ K_3-minor-free graphs

Minors



Is  a minor of  ?

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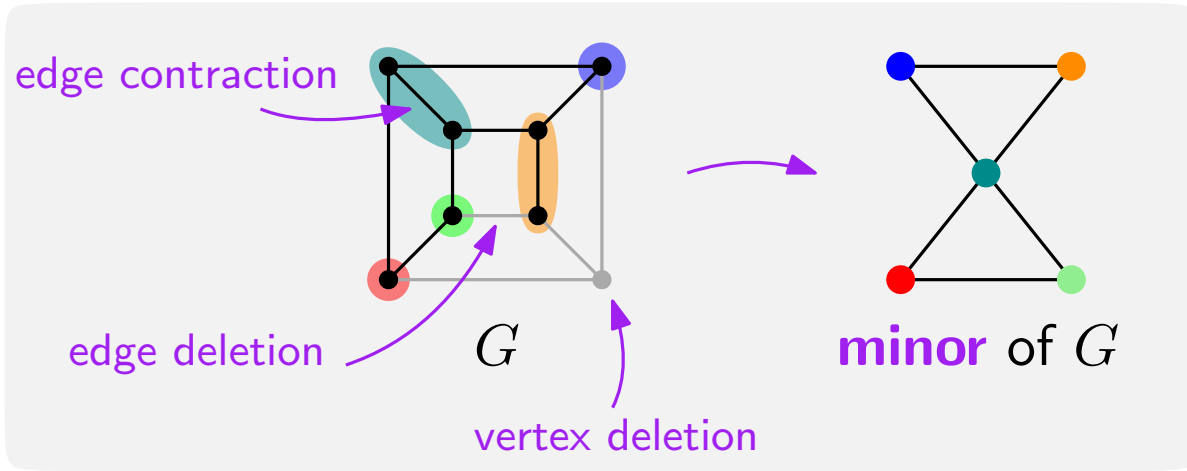
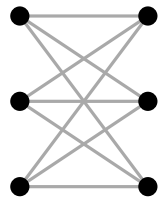
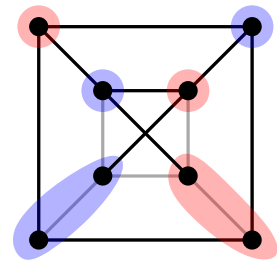


Is  a minor of  ? yes

Minor-closed graph classes

A graph class $\mathcal{H}$ is **minor-closed** if all minors of graphs in $\mathcal{H}$ are also in $\mathcal{H}$.

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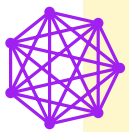

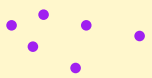

- edgeless graphs
- cliques
- trees

- forests
- bipartite graphs

- planar graphs
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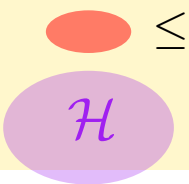

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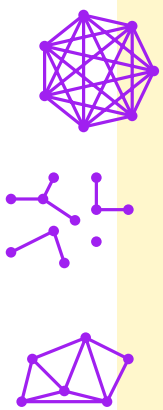
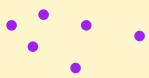


- edgeless graphs  yes
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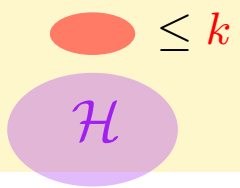

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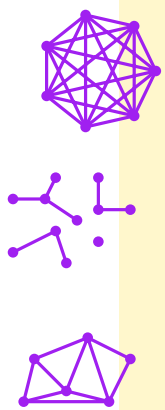

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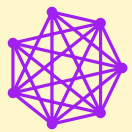
- cliques no 
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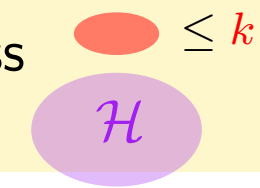

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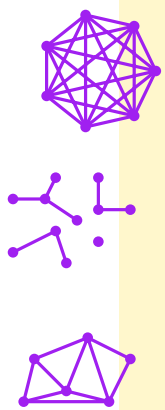

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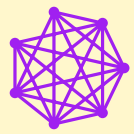
- cliques no 
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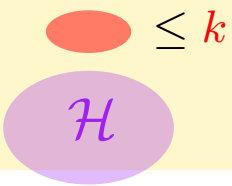

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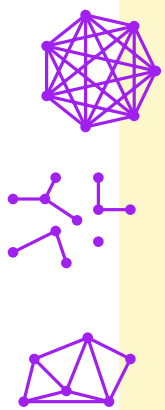

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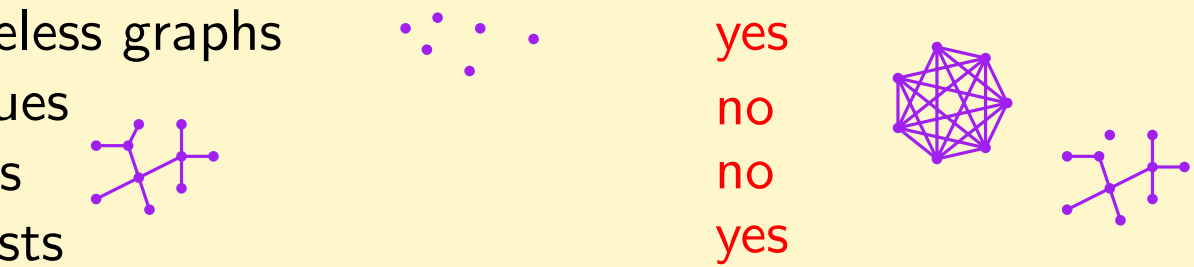

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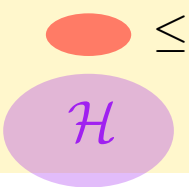

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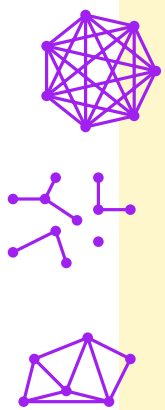

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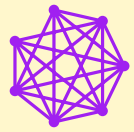
- cliques no 
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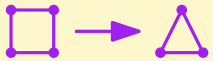
- bipartite graphs  no 
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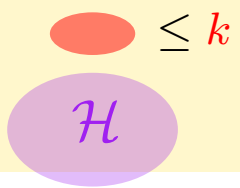

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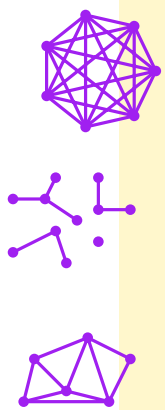

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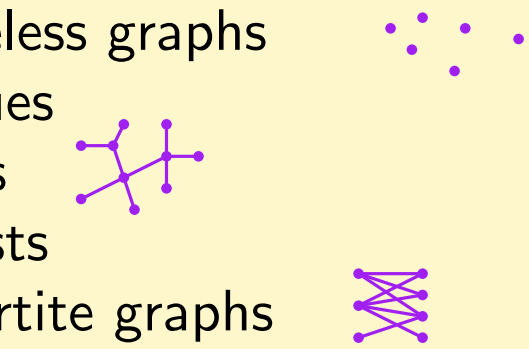

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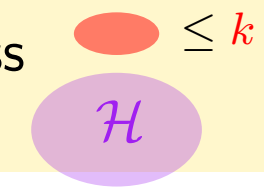

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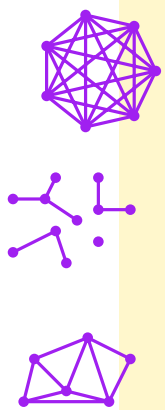

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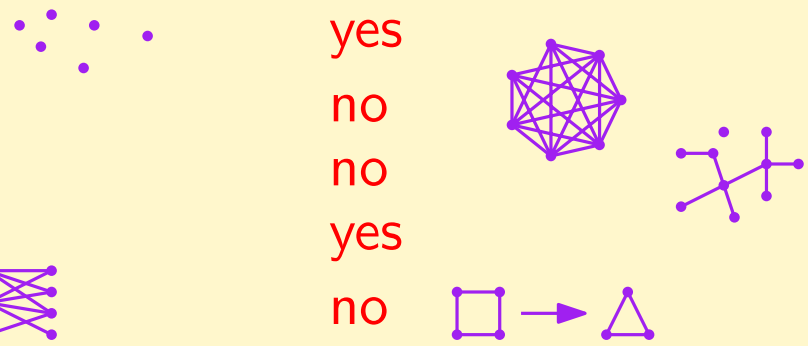

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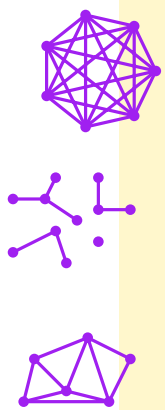

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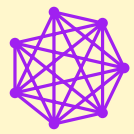
- cliques no 
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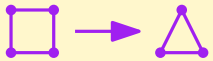
- bipartite graphs  no 
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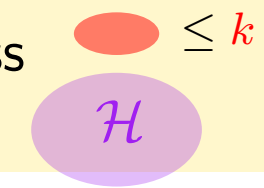
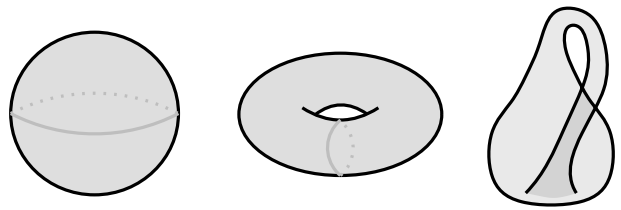


Other minor-closed graph classes:

- outerplanar graphs
- graphs embeddable on a surface



Why Graph Minor theory?

Why Graph Minor theory?

because of

the Graph Minors Series of Robertson and Seymour

23 papers, over more than 20 years (1983-2012)

Introduce important results and most of the techniques used in Graph Minor theory.

Treewidth,
the **Courcelle's theorem**,
the **Grid Exclusion theorem**,
and the **Graph Minor structure theorem**

[Graph Minors II & V & XVI], [Courcelle, 1990]

[Graph Minors II] (Re)introduction of the **treewidth**

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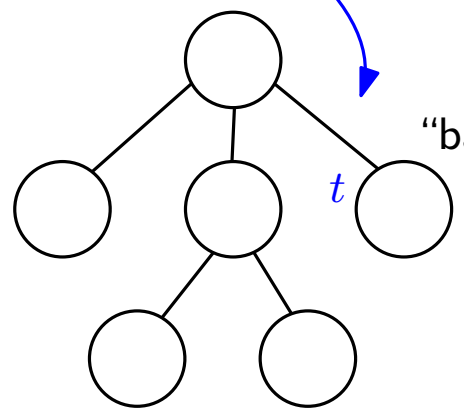
Tree decomposition $(T, \{B_t\}_{t \in V(T)})$ of G



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$B_t \subseteq V(G)$

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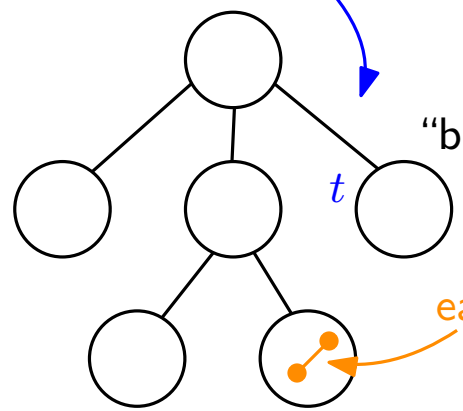

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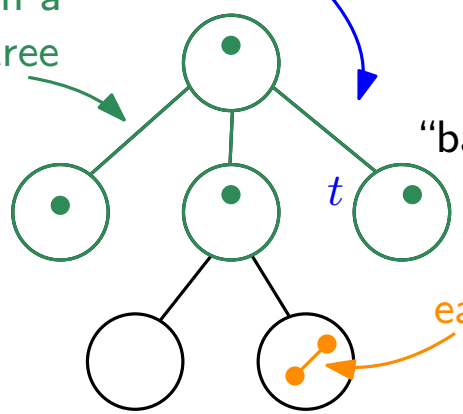

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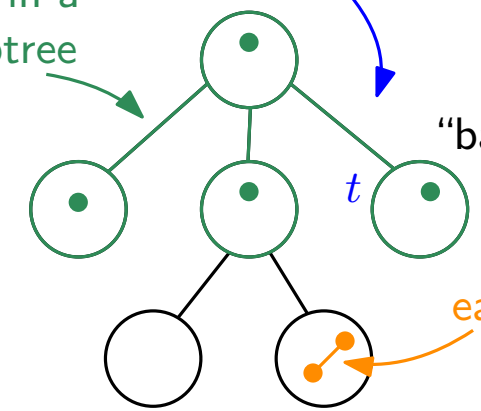

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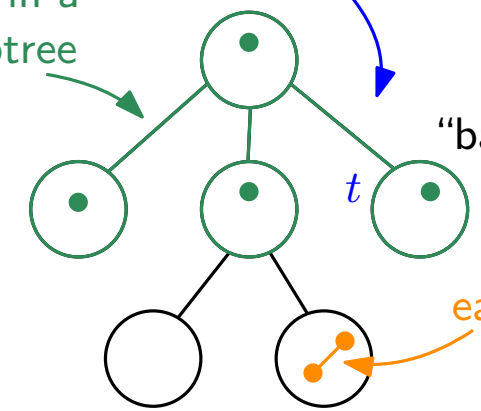

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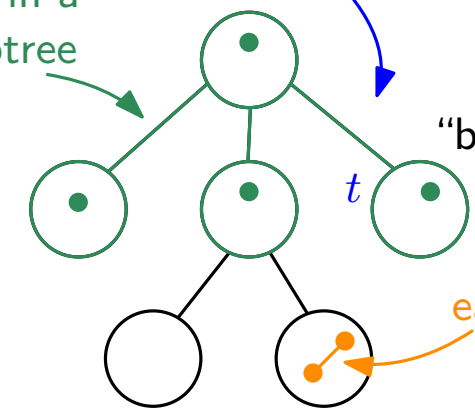

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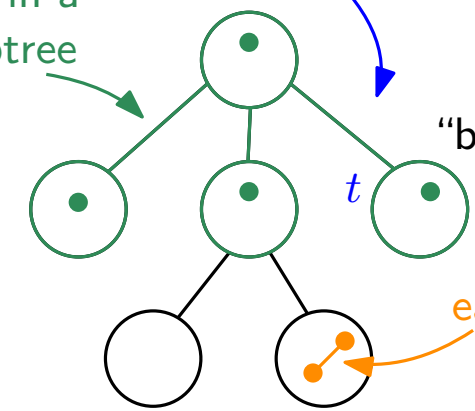

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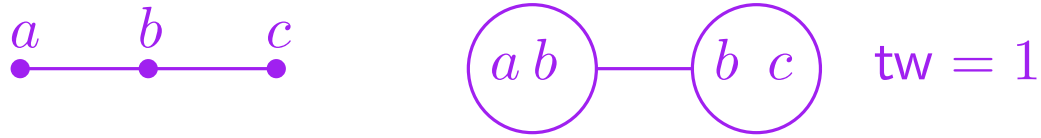

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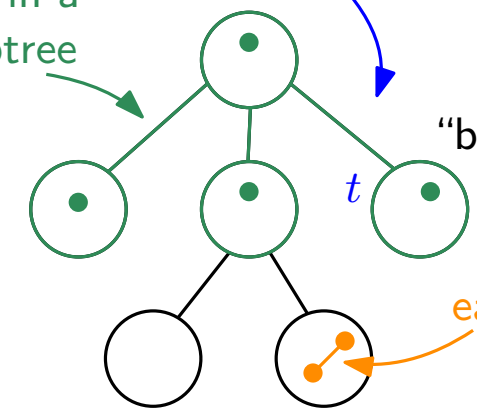

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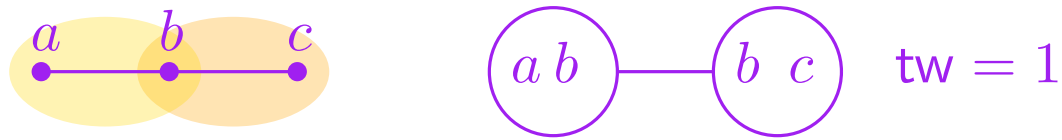

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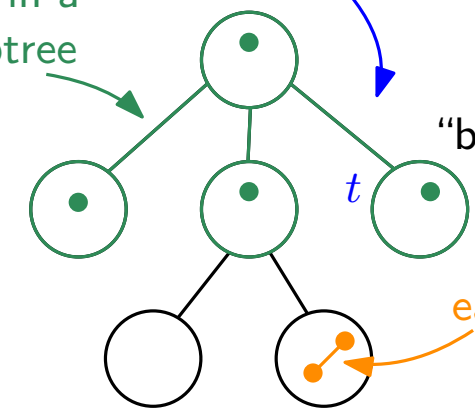

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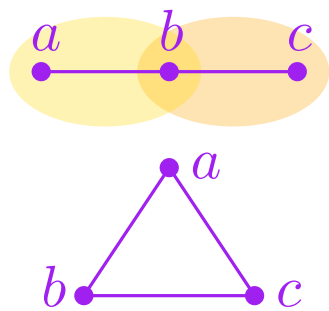

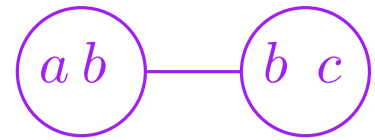

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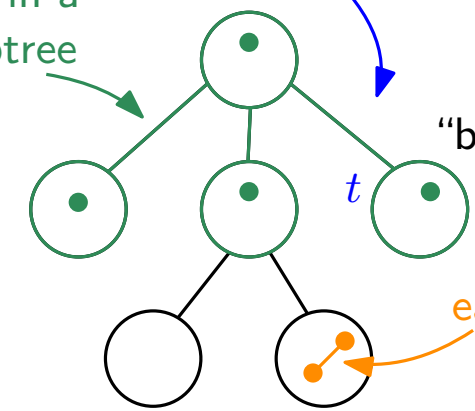

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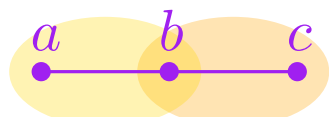

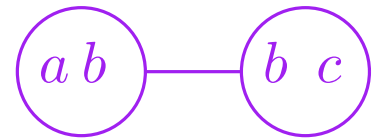

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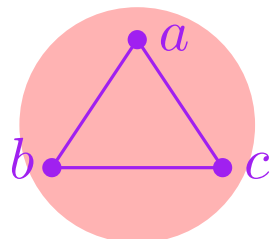

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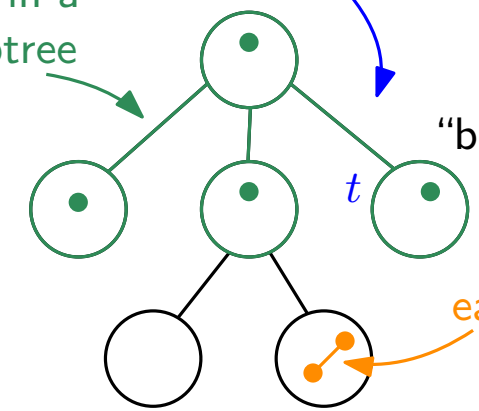

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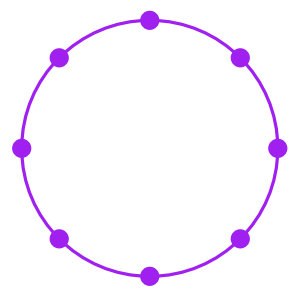

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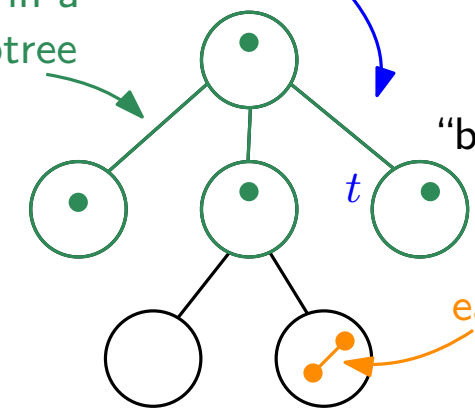

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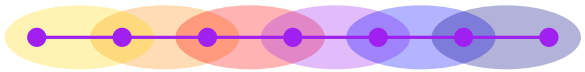

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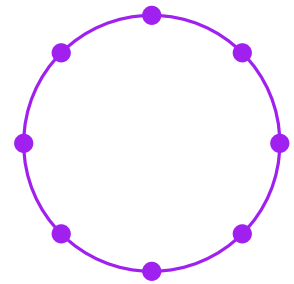

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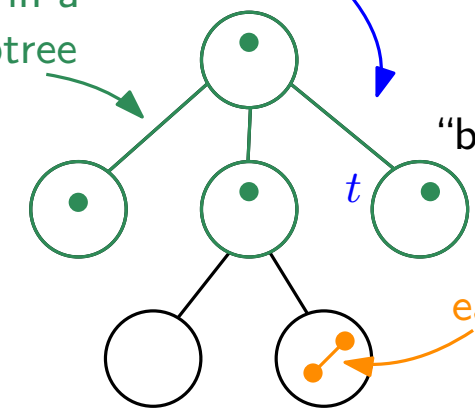

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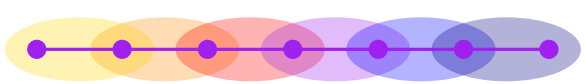

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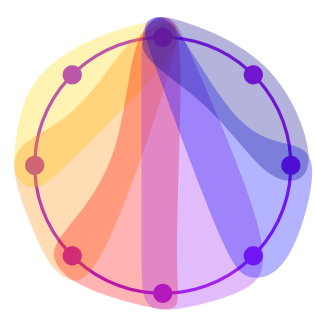

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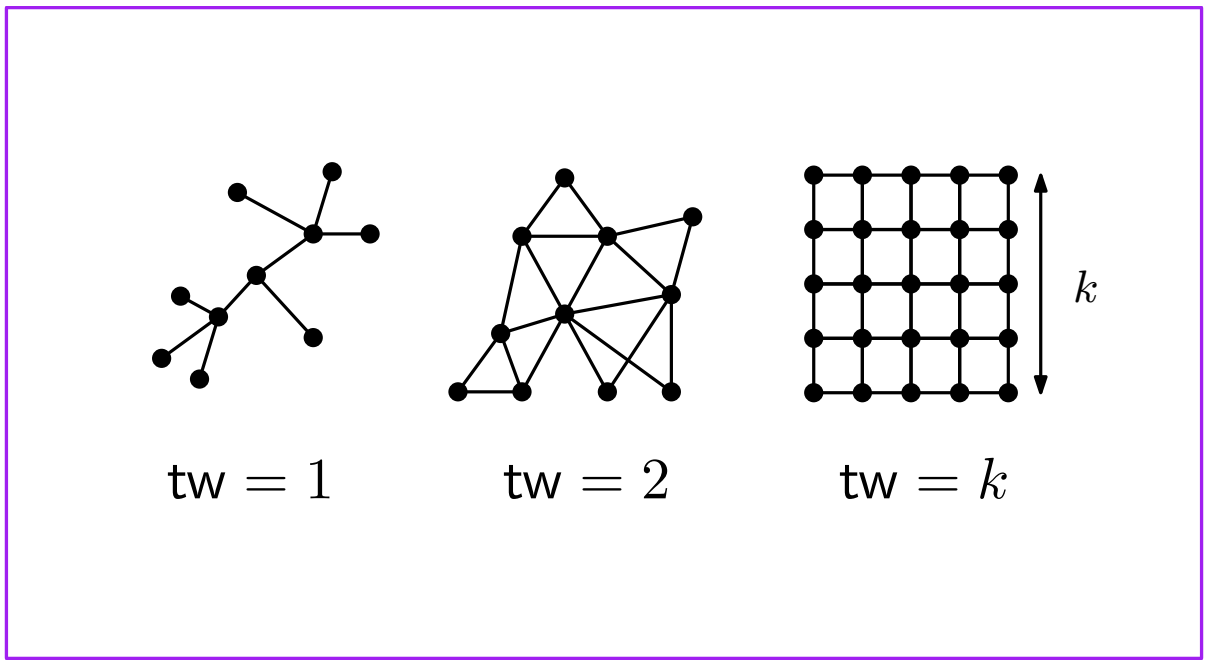

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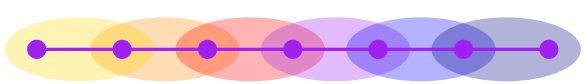

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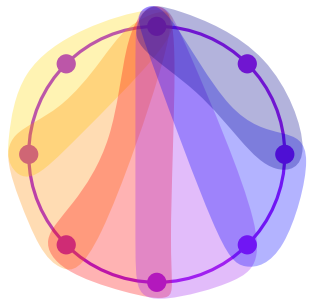

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Uses of treewidth

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First-order (FO) logic:

Variables: vertex v, edge e

$\exists, \forall, \wedge, \vee, \neg, =, \mathsf{inc}(\cdot,\cdot), \mathsf{adj}(\cdot,\cdot)$

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Property of containing a triangle as a subgraph:

$\exists v_1, \exists v_2, \exists v_3, (\mathsf{adj}(v_1,v_2) \wedge (\mathsf{adj}(v_2,v_3)) \wedge (\mathsf{adj}(v_1,v_3))$

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How to express the property of having a vertex cover of size 2?

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Monadic Second-Order (MSO_2) logic:

extension of FO logic with:

New variables: vertex set V, edge set E

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How to express connectivity?

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$\exists, \forall, \wedge, \vee, \neg, =, \mathsf{inc}(\cdot,\cdot), \mathsf{adj}(\cdot,\cdot)$

How to express the property of having a vertex cover of size 2?

$\exists v_1, \exists v_2, \forall e, (\mathsf{inc}(v_1, e) \vee (\mathsf{inc}(v_2, e))$

$\mathsf{inc}(v, e) = e$ is incident to v, $\mathsf{adj}(v_1, v_2) = v_1$ is adjacent to v_2

Monadic Second-Order (MSO_2) logic:

extension of FO logic with:

New variables: vertex set V, edge set E

New relation: $\in$

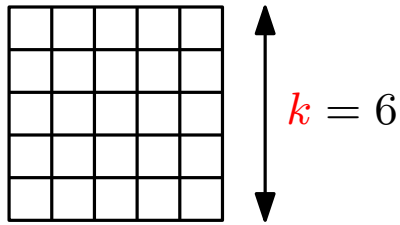
How to express 2-colorability?

$\exists V, \forall v_1, \forall v_2, \neg(\mathsf{adj}(v_1, v_2) \wedge$
$((v_1 \in V \wedge v_2 \in V) \vee (\neg v_1 \in V \wedge \neg v_2 \in V)))$

How to express connectivity?

Counting Monadic Second-Order (CMSO) logic: extension of MSO_2 with $|\cdot| = q \mod r$ $(r \geq 2)$

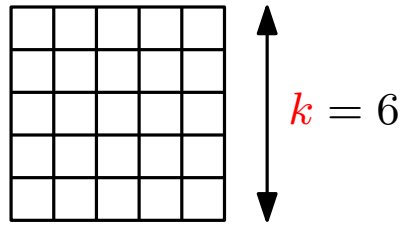
[Graph Minors V] Grid Exclusion theorem



Grid Exclusion theorem:

$\exists g$ s.t., if a graph does not contain a grid of height k as a minor, then it has treewidth $\leq g(k)$.

[Graph Minors V] Grid Exclusion theorem



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An application: Win/Win strategy for Feedback Vertex Set

Input: G, k.
Question: Is there a vertex set S of size $\leq k$ such that $G - S$ is a forest?

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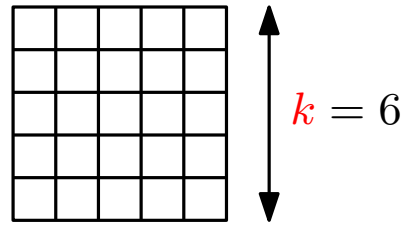

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An application: Win/Win strategy for Feedback Vertex Set

- If G contains a grid of heigth $2k$ as a minor
  $\rightarrow$ no-instance.
- Otherwise, G has treewidth $\leq g(k)$
  $\rightarrow$ conclude using Courcelle's theorem.

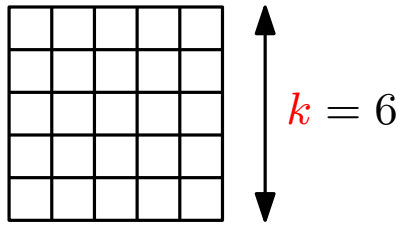
Input: G, k.
Question: Is there a vertex set S of size $\leq k$ such that $G - S$ is a forest?

$\rightarrow$ Solve FVS in FPT-time parameterized by k.

Courcelle's theorem

Every problem expressible in CMSO logic is solvable in time $f(k) \cdot n$ on graphs of treewidth $\leq k$.

[Graph Minors V] Excluding a planar graph



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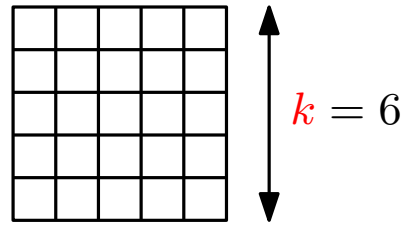

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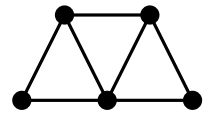
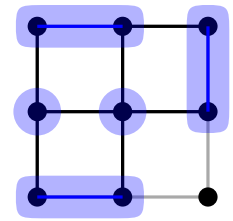
Observation:

Every planar graph is a minor of a big enough grid.





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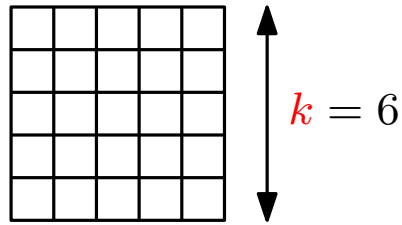

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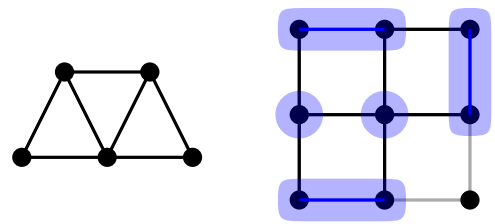
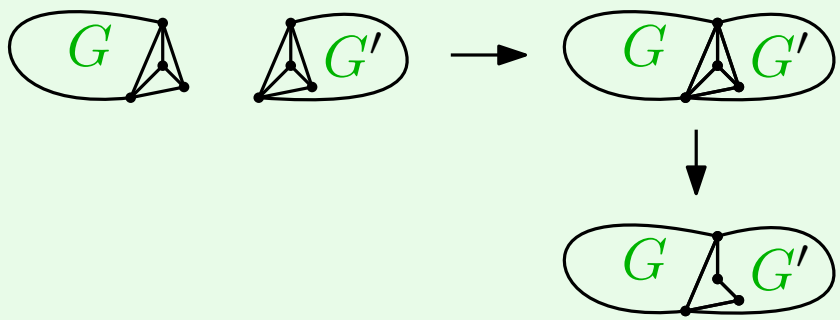


Equivalent definition of the treewidth:

A **clique-sum** of G and G':
- identify a clique of G with a clique of G'
- can delete some edges of the clique



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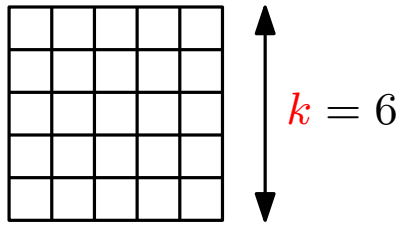

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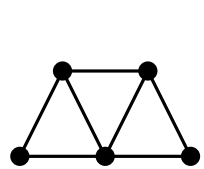

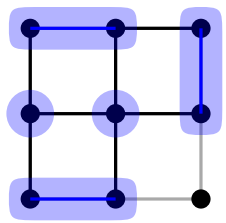

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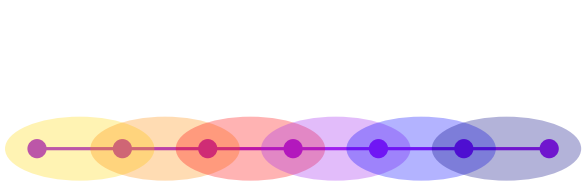

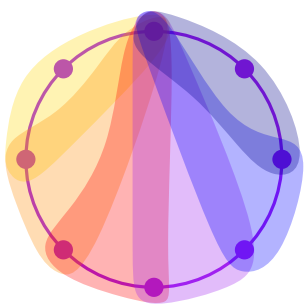

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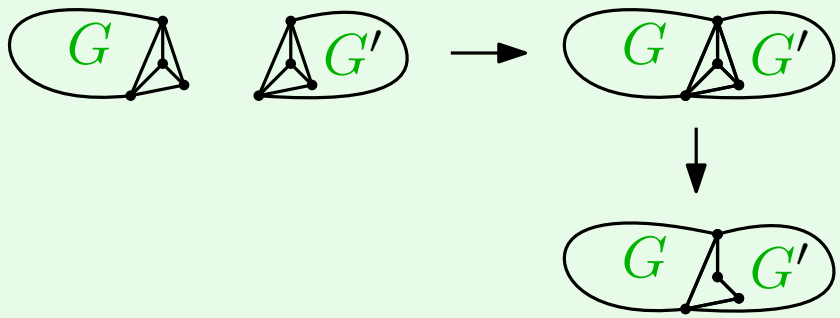

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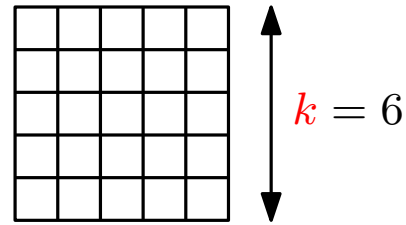

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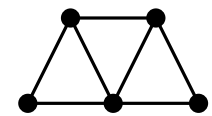

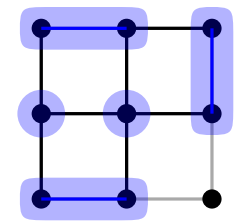

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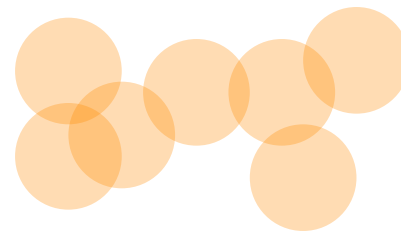


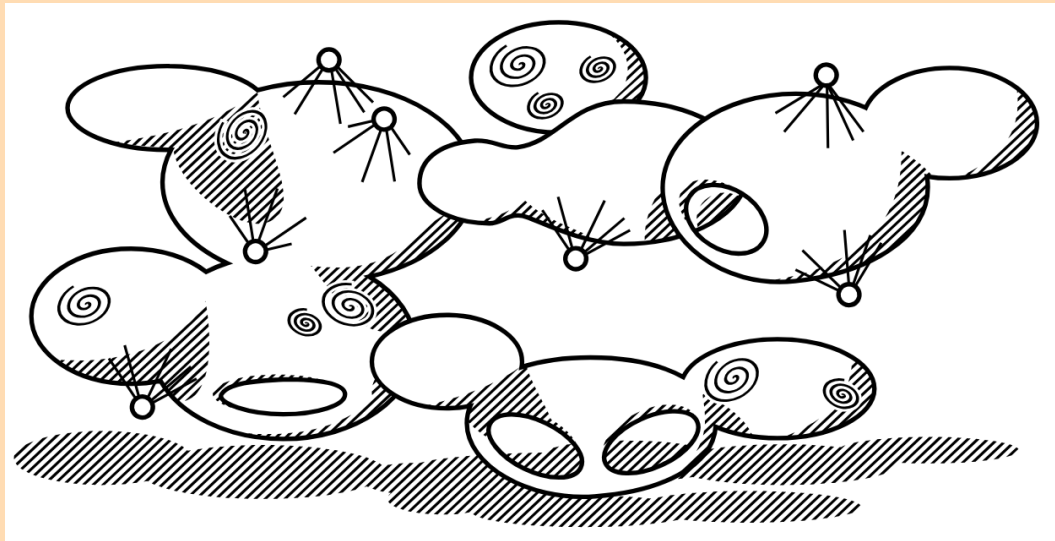
Structure theorem (planar case)

If a graph G excludes a planar graph H as a minor, then it is a clique-sum of graphs of size $\leq c_H$.

[Graph Minors XVI] Excluding a non-planar graph

Graph Minor Structure theorem

If a graph G excludes a graph H as a minor, then it looks like this:



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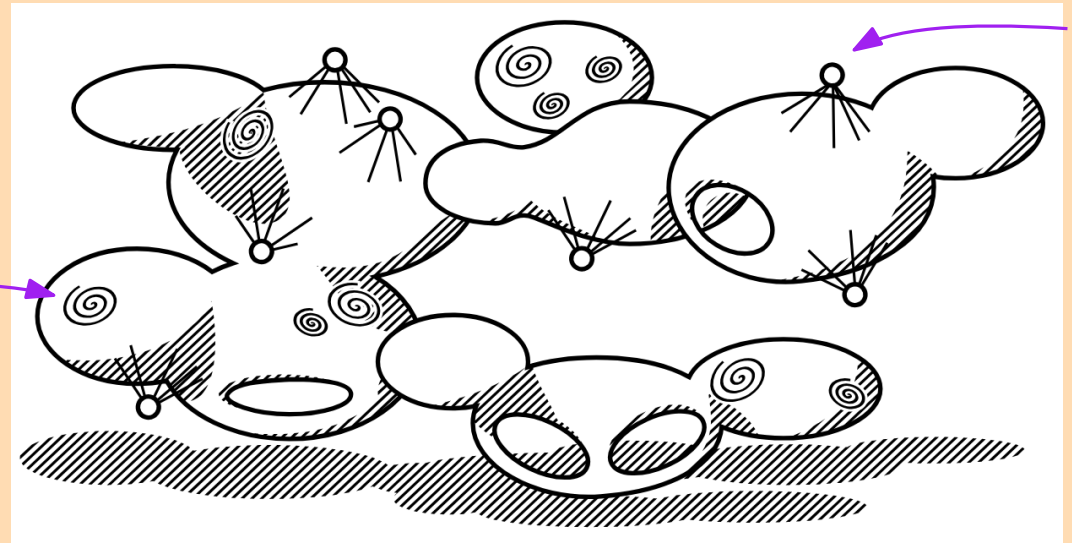

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i.e. G is a clique-sum of graphs that, after removing $\leq a_H$ apices, are embeddable in a surface of genus $\leq g_H$, aside from $\leq b_H$ regions of "width" $\leq w_H$ called vortices.

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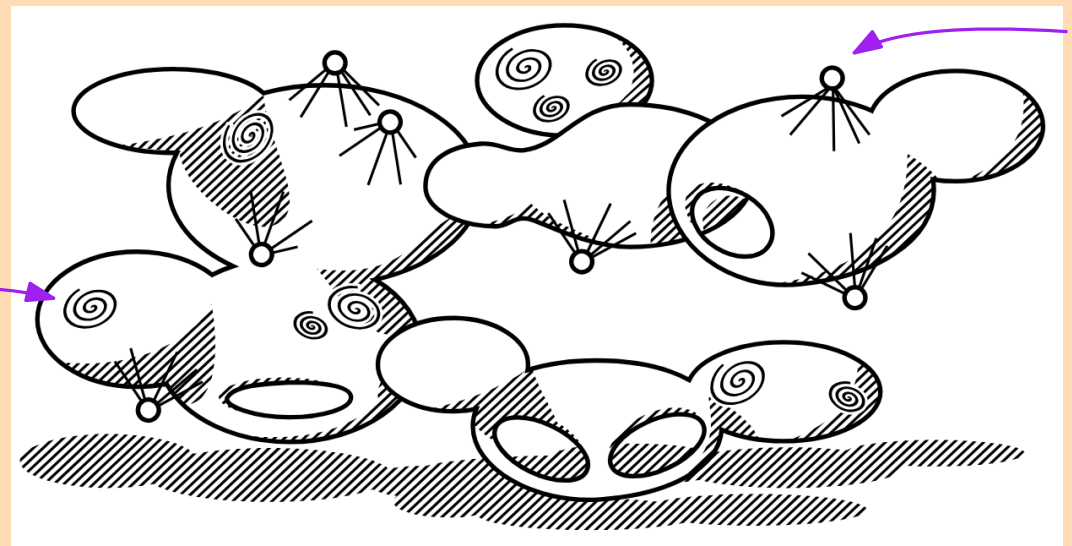

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If know how to solve a problem on planar graphs, can then try to solve the problem on bounded genus graphs, and then on minor-closed graph classes.

The **Irrelevant Vertex technique**

and the **Unique Linkage theorem**
and the **Flat Wall theorem**

[Graph Minors XIII & XXI]

[Graph Minors XIII] The **Irrelevant Vertex technique**

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A very powerful technique for problems related to minor-closed graph classes...

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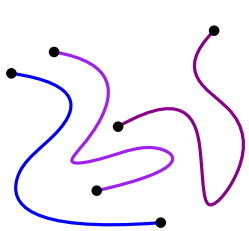
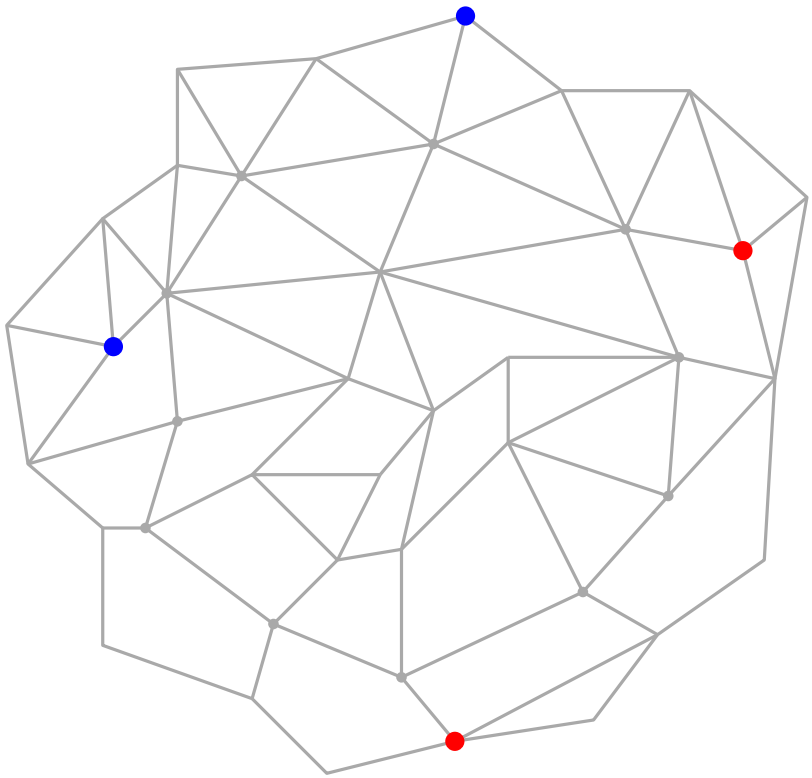
Example of the k-Disjoint Paths problem (kDPP)



Input: A graph G and terminals $(s_1, t_1), \ldots, (s_k, t_k)$.
Questions: Is there k disjoint paths $P_1, \ldots, P_k$ s.t. P_i is a s_i-t_i path for $1 \leq i \leq k$?

Easy case: 2DPP on planar graphs



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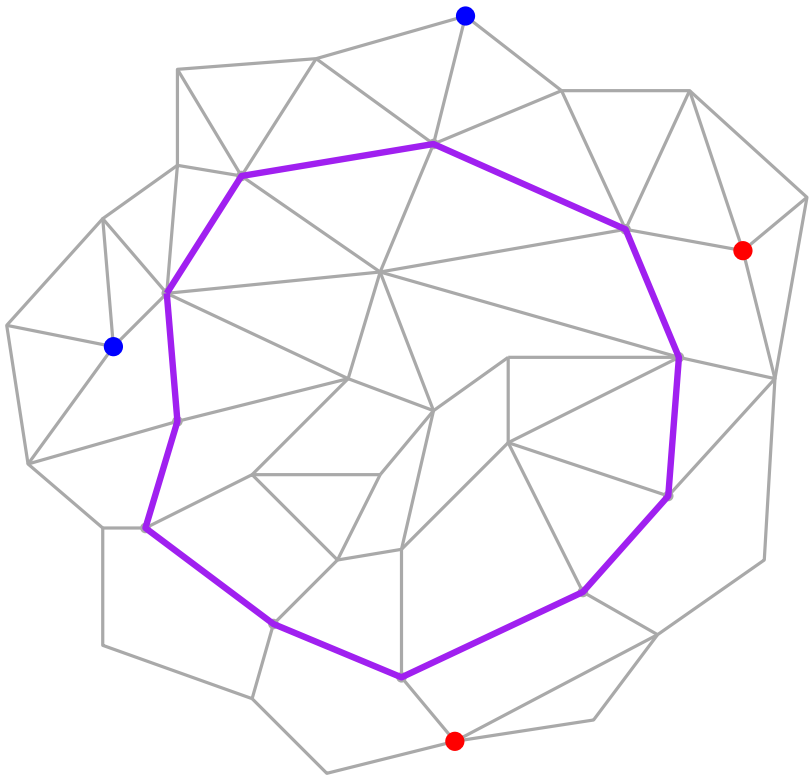

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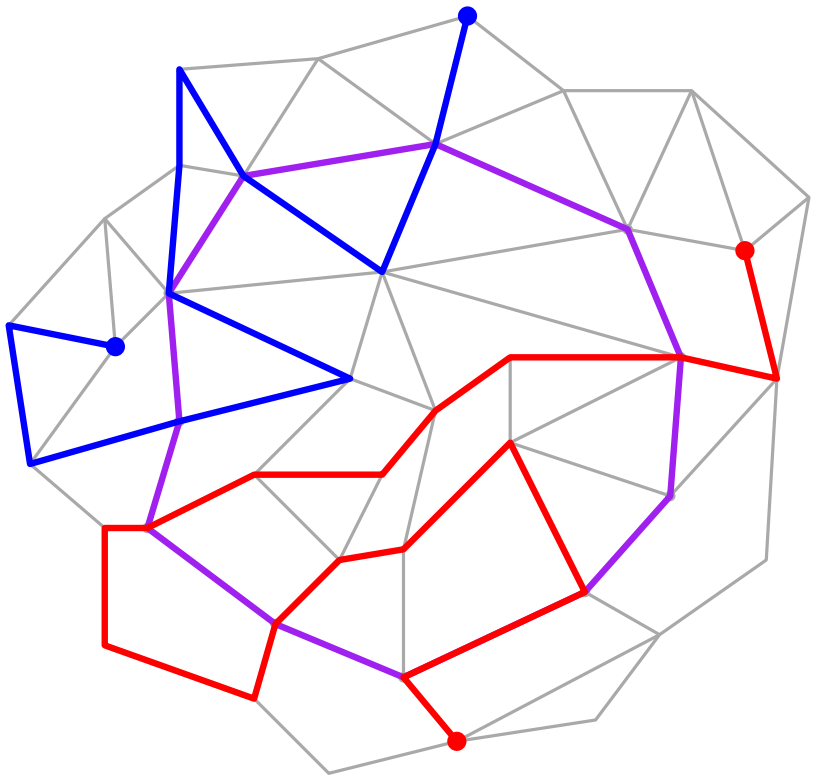

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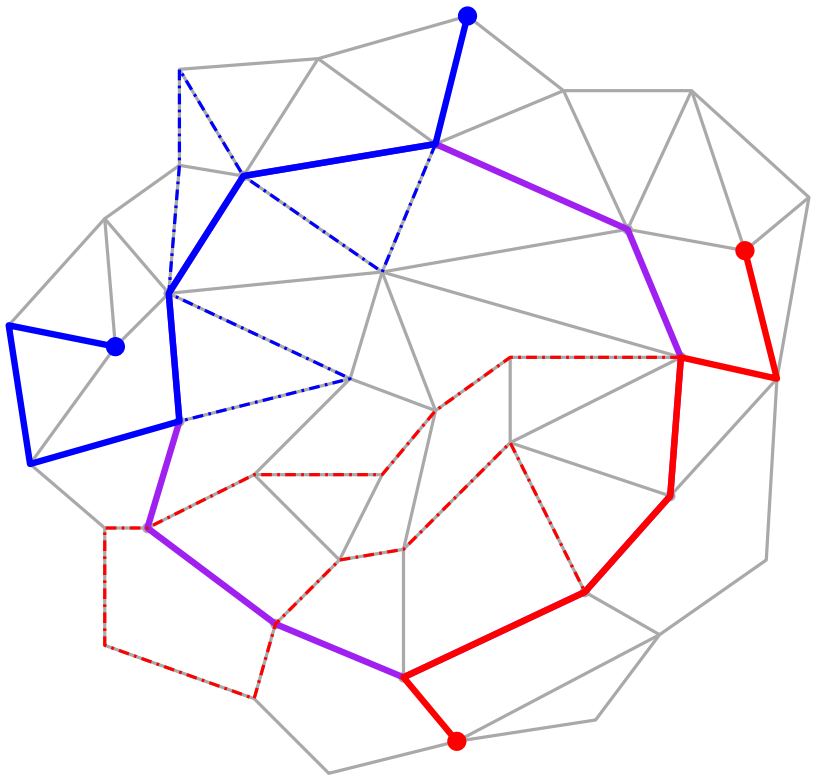

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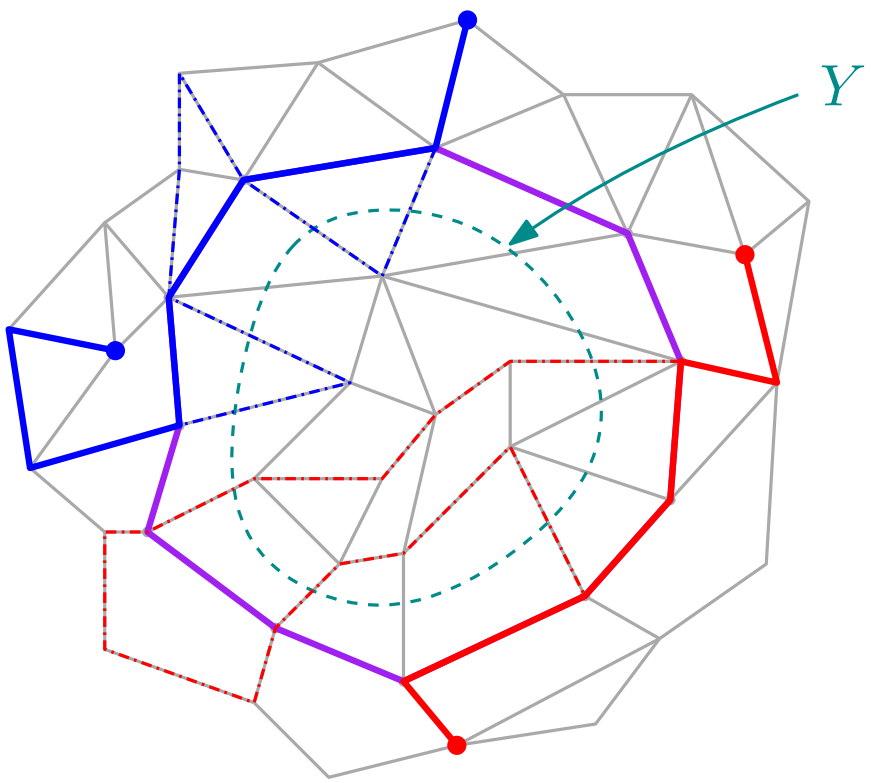

If $G - Y$ is a yes-instance of 2DPP,
then G is a yes-instance of 2DPP

because the 2 disjoint paths of $G - Y$ are also disjoint paths of G.

If G is a yes-instance of 2DPP,
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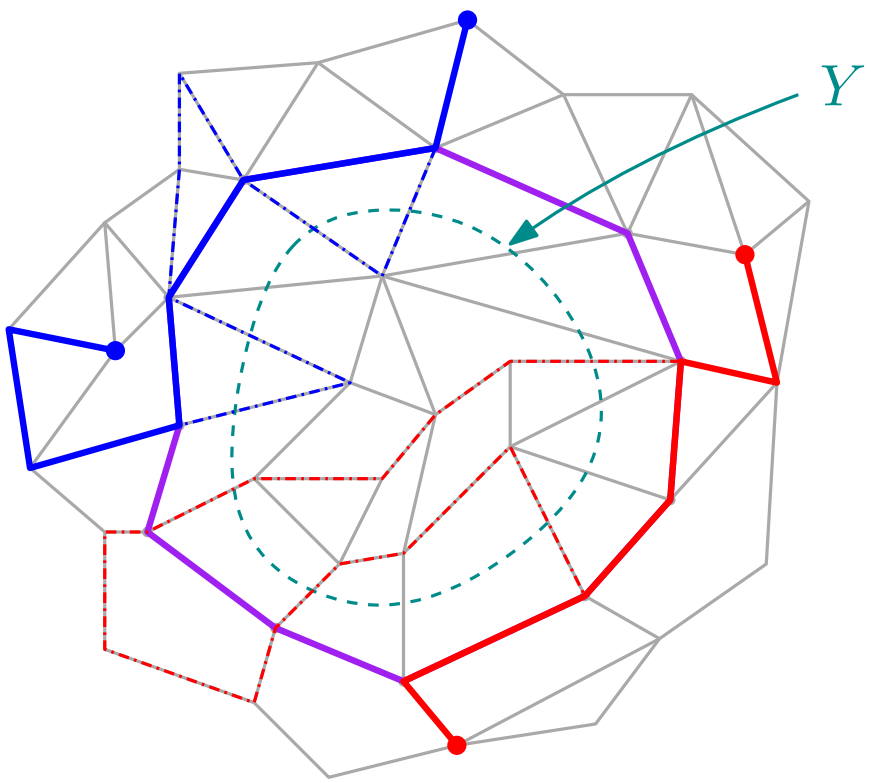

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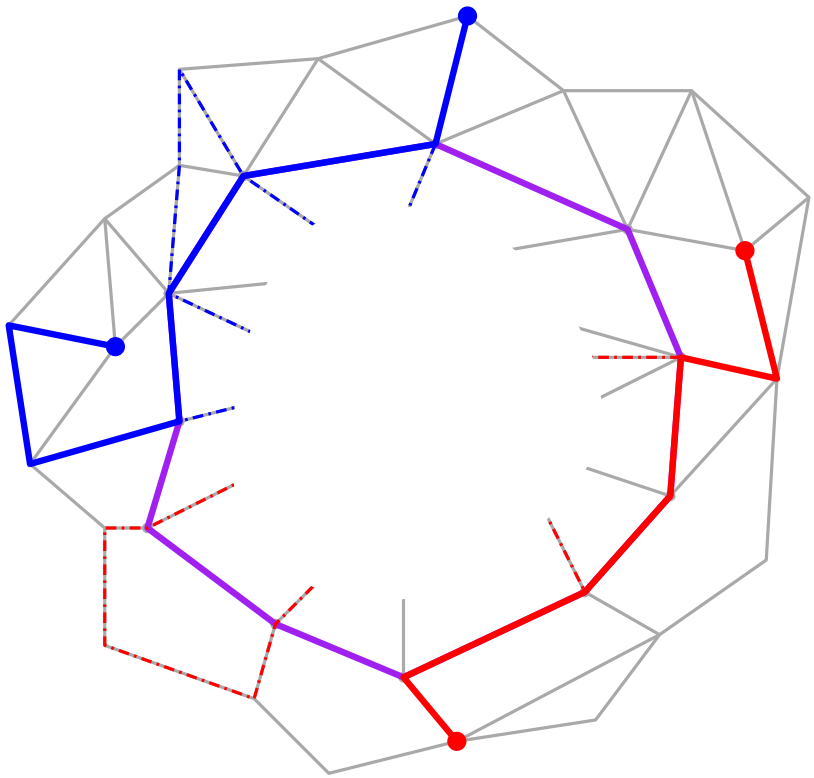
$\Longrightarrow$ G and $G - Y$ are equivalent instances of 2DPP.

The vertices of Y are **irrelevant**.

Easy case: 2DPP on planar graphs



$\Longrightarrow$ can delete Y and recurse.

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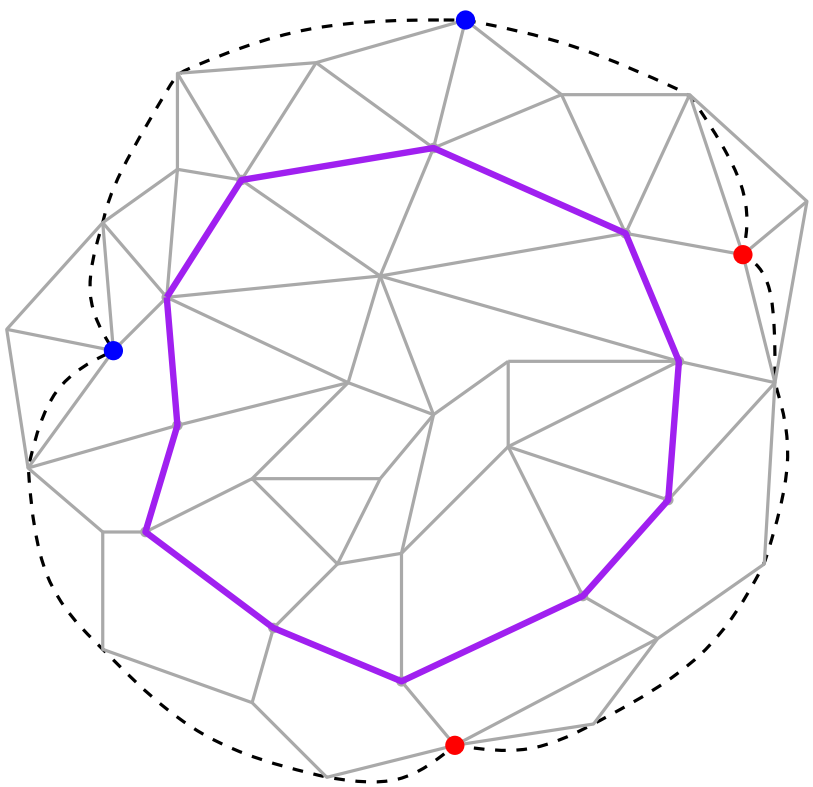
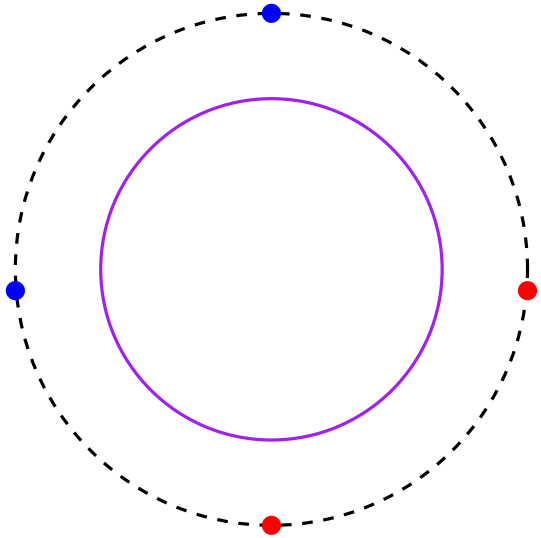


Simplified representation:



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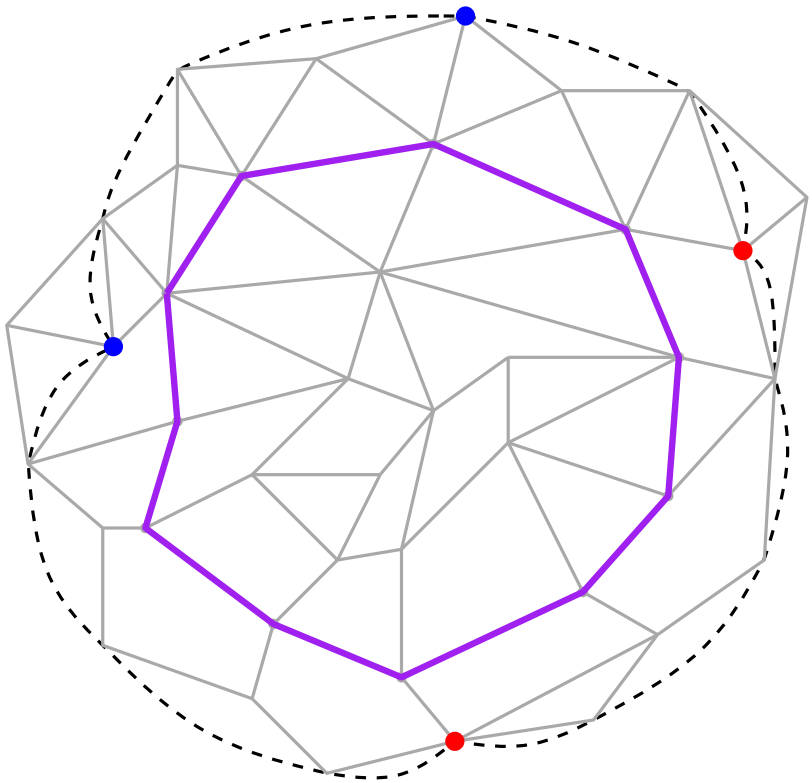

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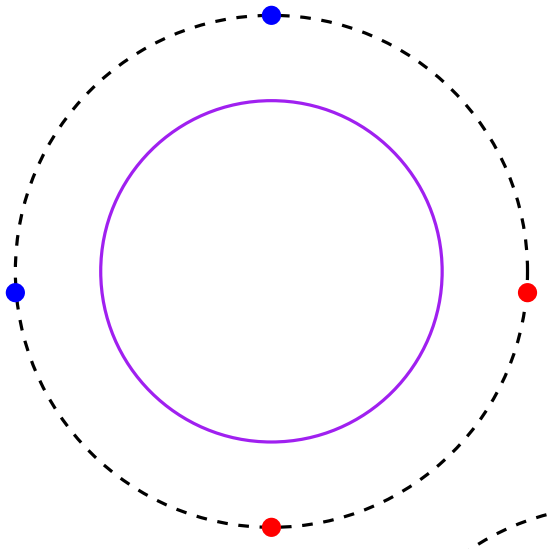

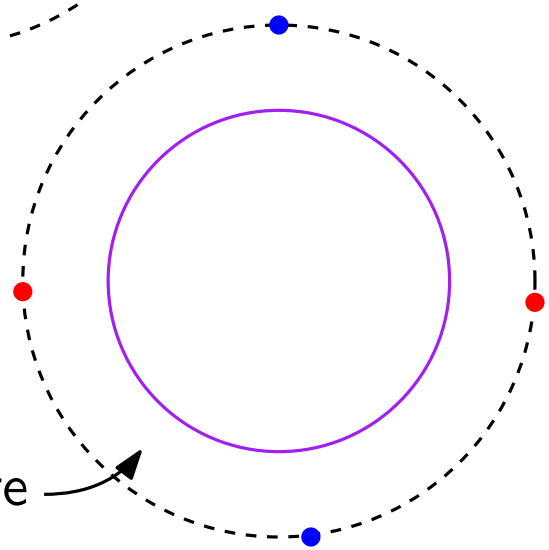

The order on the boundary matters!

no disjoint paths here

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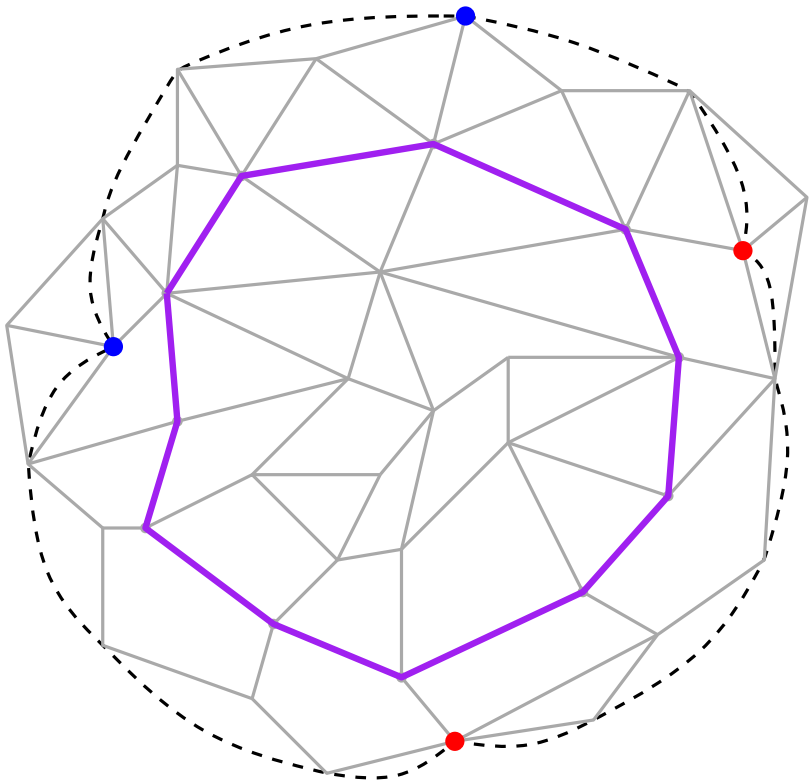

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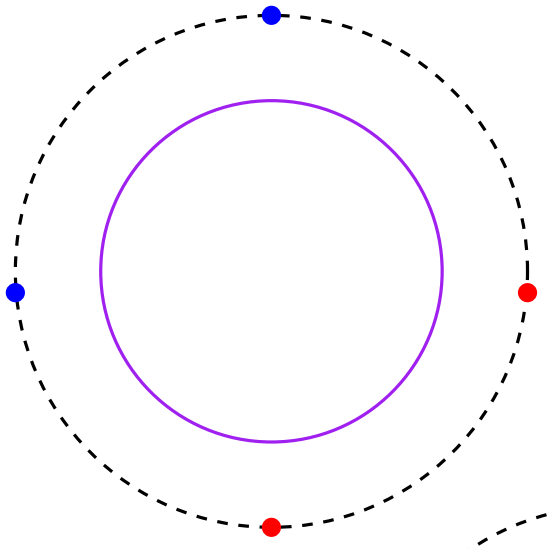

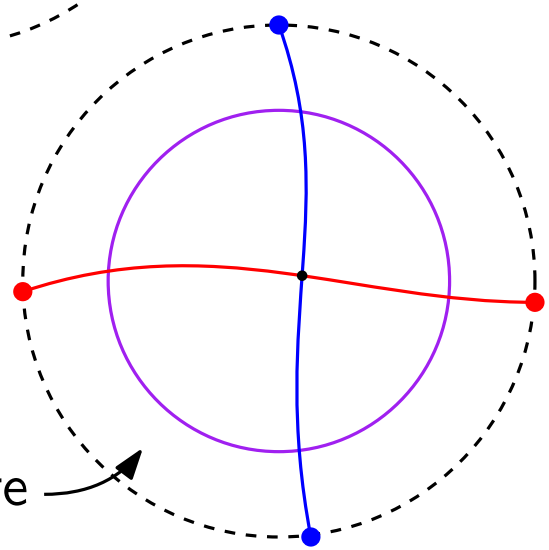

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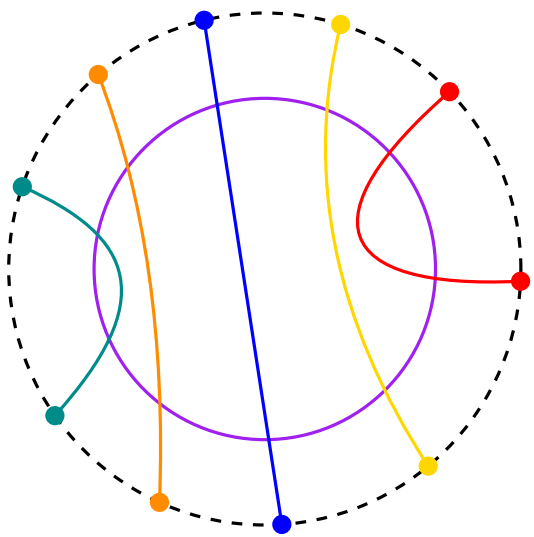
Slightly more difficult case: kDPP on planar graphs



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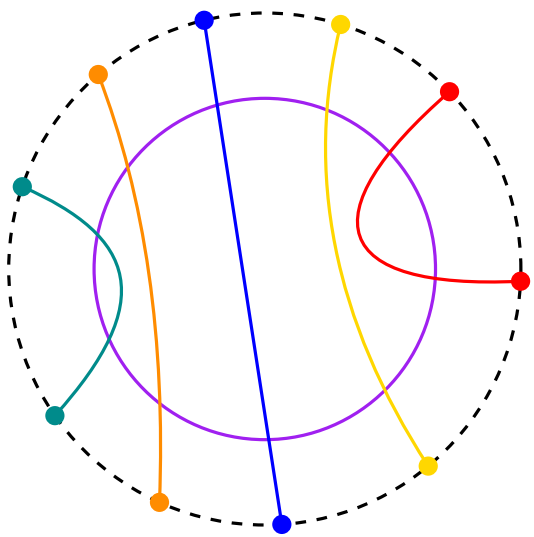
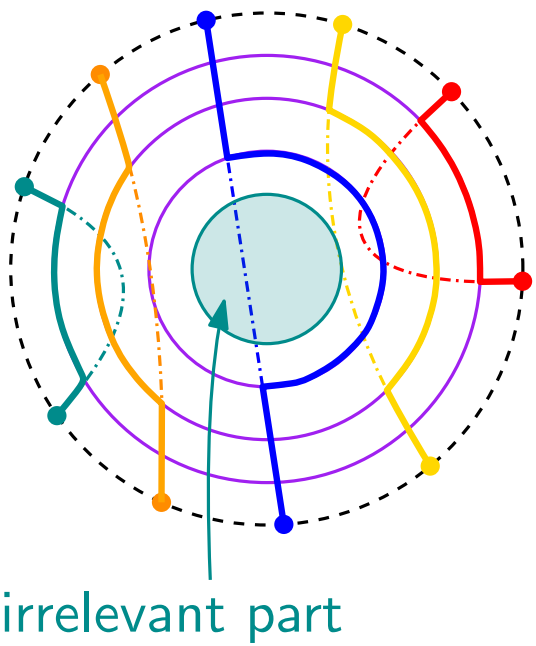


One cycle is not enough to reroute paths.

Slightly more difficult case: kDPP on planar graphs



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[Graph Minors XXI]

Unique Linkage theorem (Corollary)

If G constains $f(k)$ nested cycles, with all terminals outside the outer cycle, then all paths of a solution of kDPP can be rerouted away from the inner cycle.

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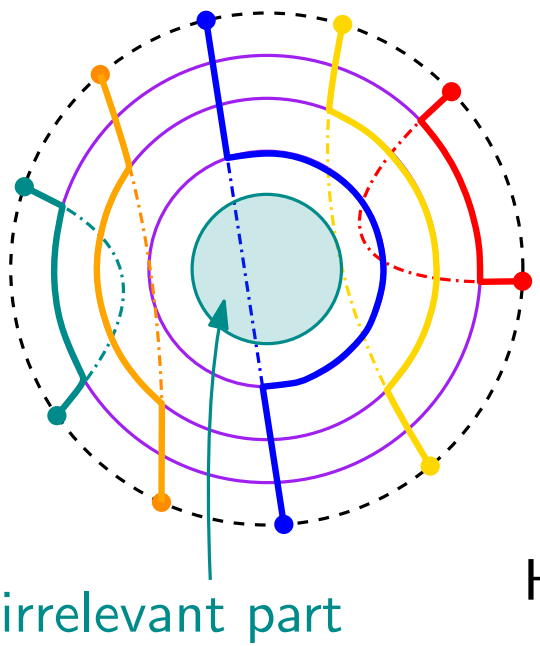

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How to solve kDPP?

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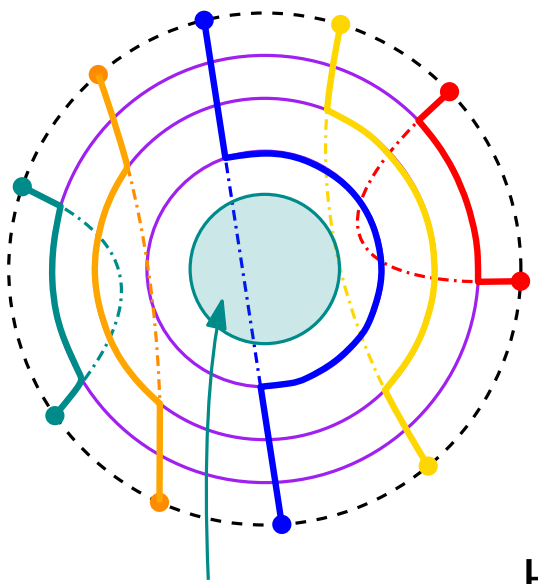

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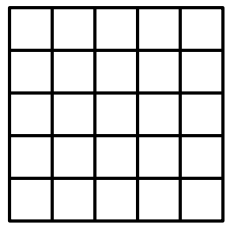
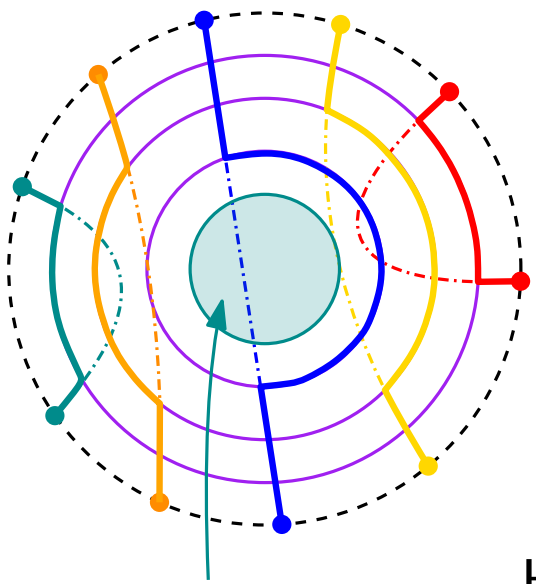


If G has treewidth $\geq f_1(k)$: (Grid Exclusion theorem)
it contains a grid of height $f_2(k)$ as a minor

Slightly more difficult case: kDPP on planar graphs



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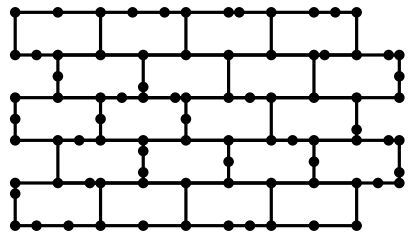
easier to find



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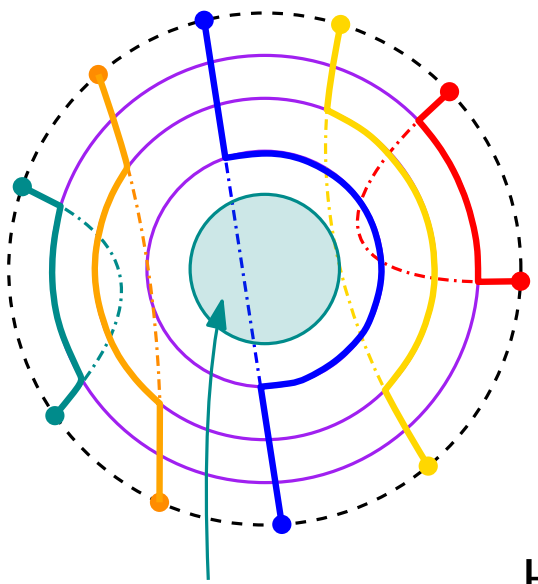

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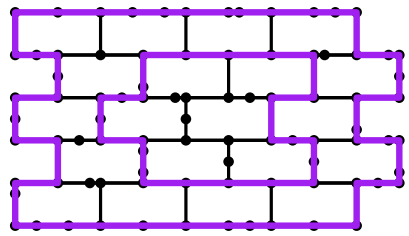

nested cycles

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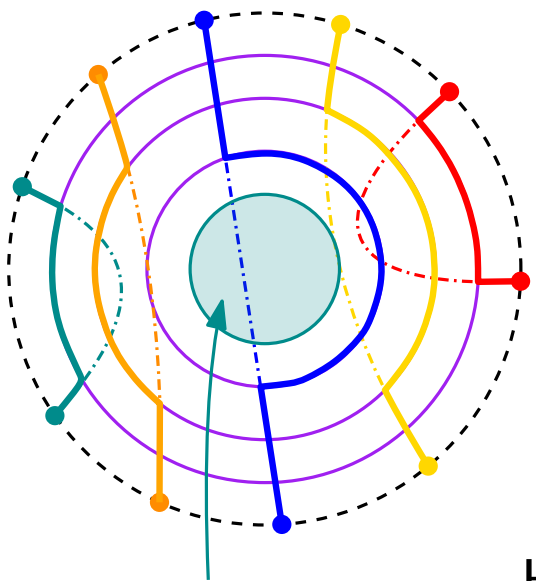

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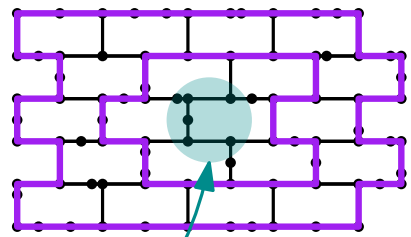

nested cycles

center of the wall is irrelevant:
delete and recurse

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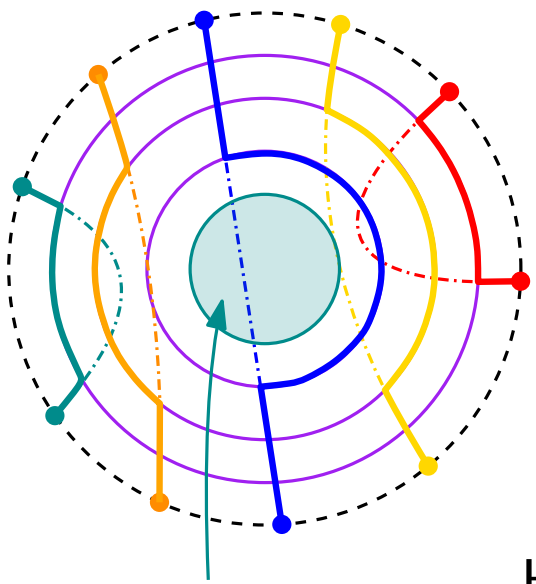

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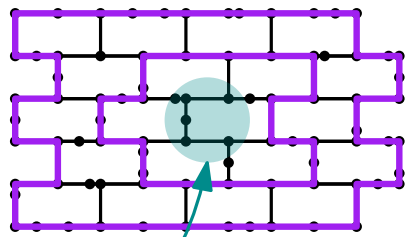

If G has treewidth $< f_1(k)$:
conclude with Courcelle's theorem

nested cycles

center of the wall is irrelevant:
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Far more difficult case: kDPP on general graphs

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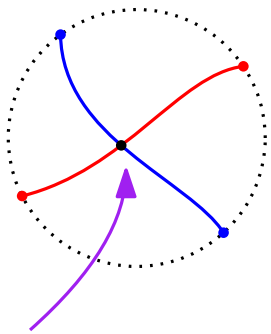
Planar case:



any 2 paths that cross one another must intersect

Far more difficult case: kDPP on general graphs

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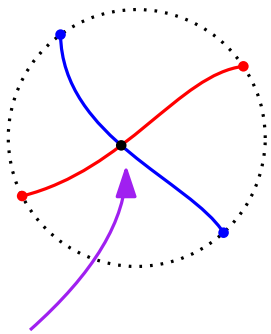

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How to get a similar property on general graphs?

Far more difficult case: kDPP on general graphs

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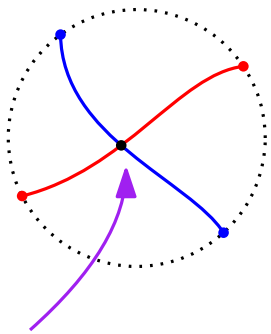

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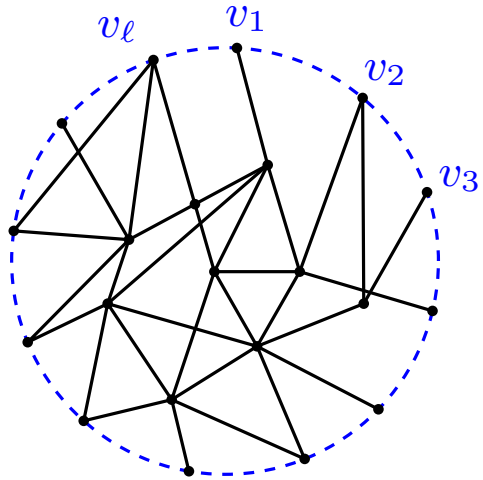

Society: pair (G, Ω)

cyclic ordering with $V(\Omega) = \{v_1, v_2, \ldots, v_\ell\}$.

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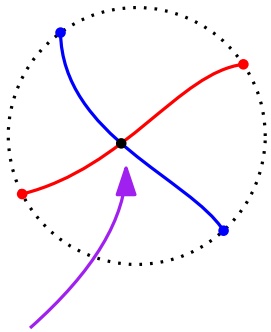

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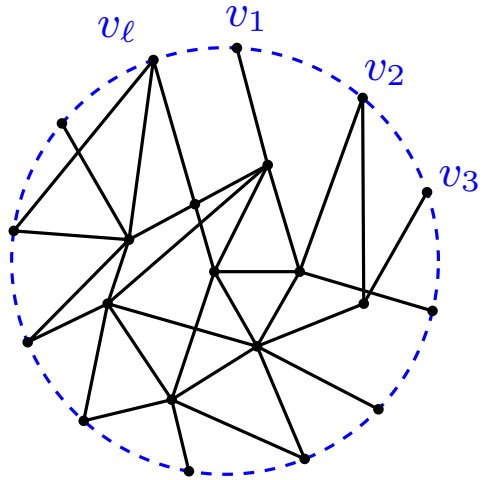

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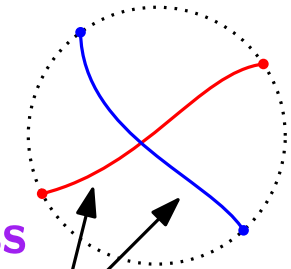
[Graph minor XIII]

A society has no cross if and only if it admits a **rural rendition**.



a **cross**

disjoint paths

Far more difficult case: kDPP on general graphs

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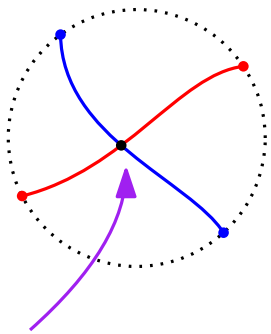

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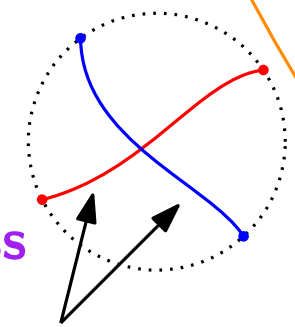

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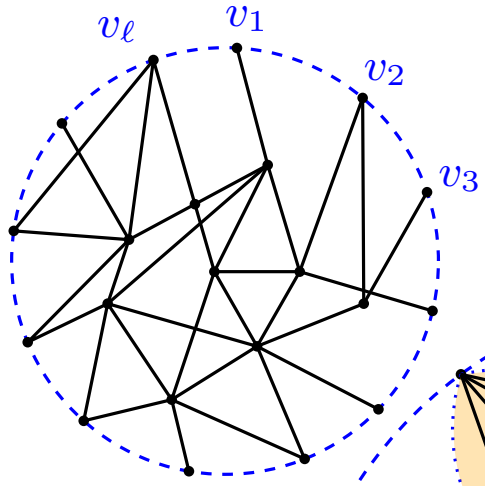

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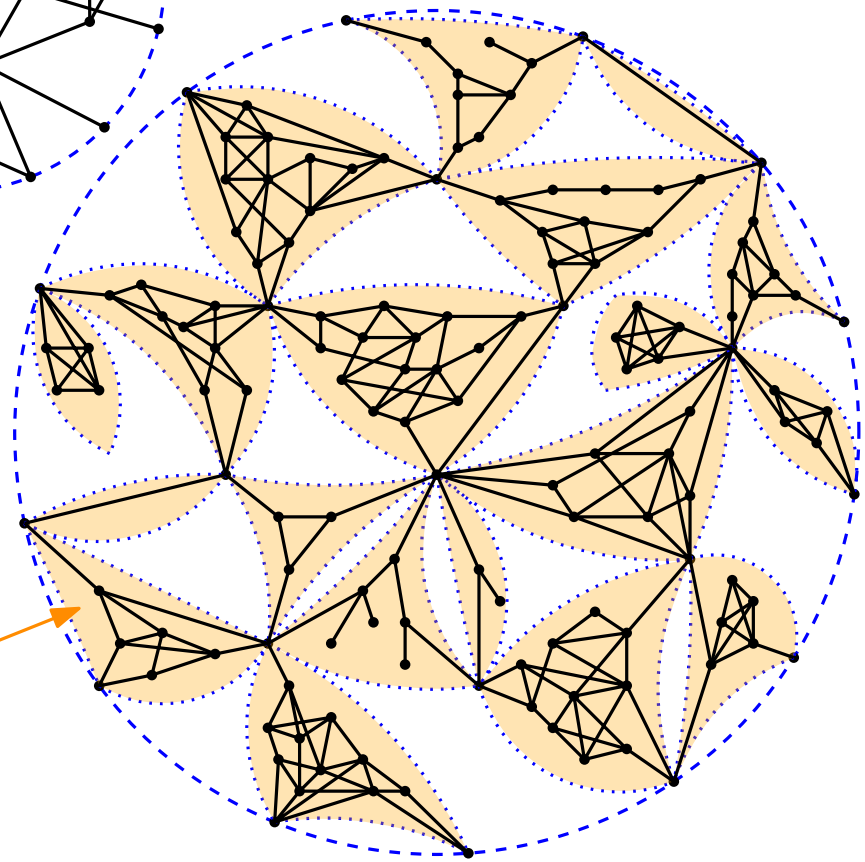

cell

boundary cell ≤ 3

Far more difficult case: kDPP on general graphs

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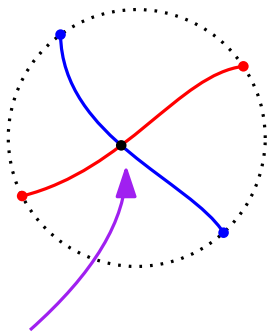

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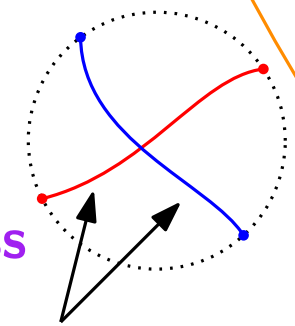

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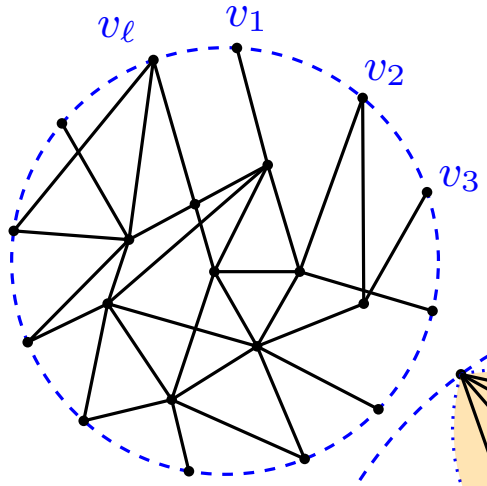

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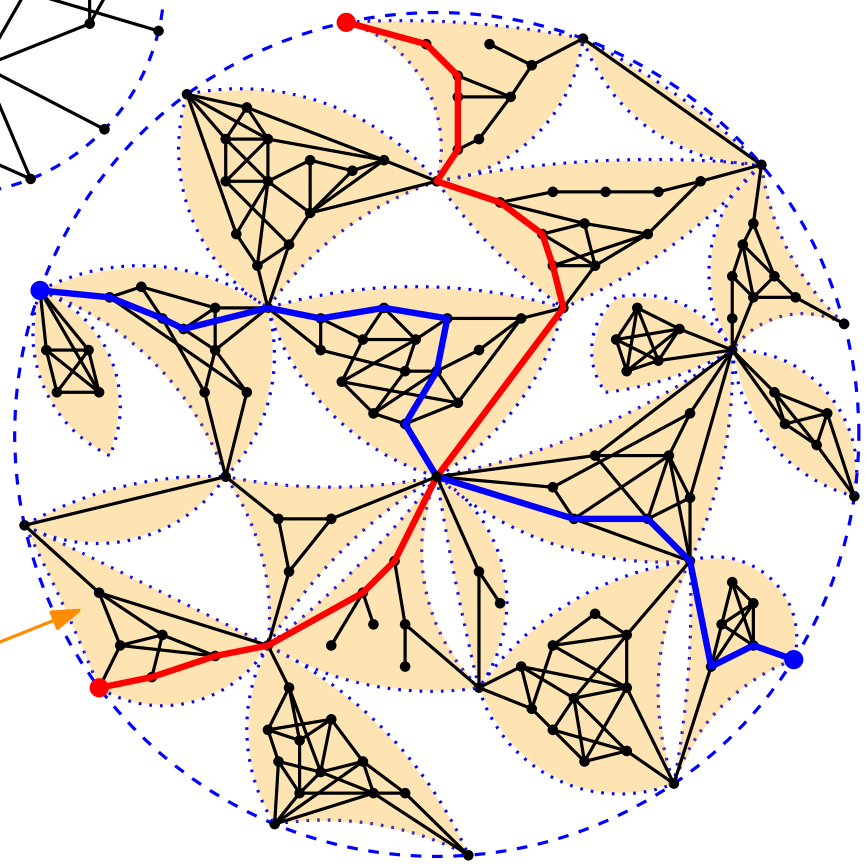

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Far more difficult case: kDPP on general graphs

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Flat Wall theorem

FPT algo in k and t

A graph G either has:

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- a clique of size t as a minor, or
- a vertex set $A \leq g(t)$ and a wall W of height k s.t. W is a **flat wall** of $G - A$.

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not covered here (can also find some kind of irrelevant vertex)

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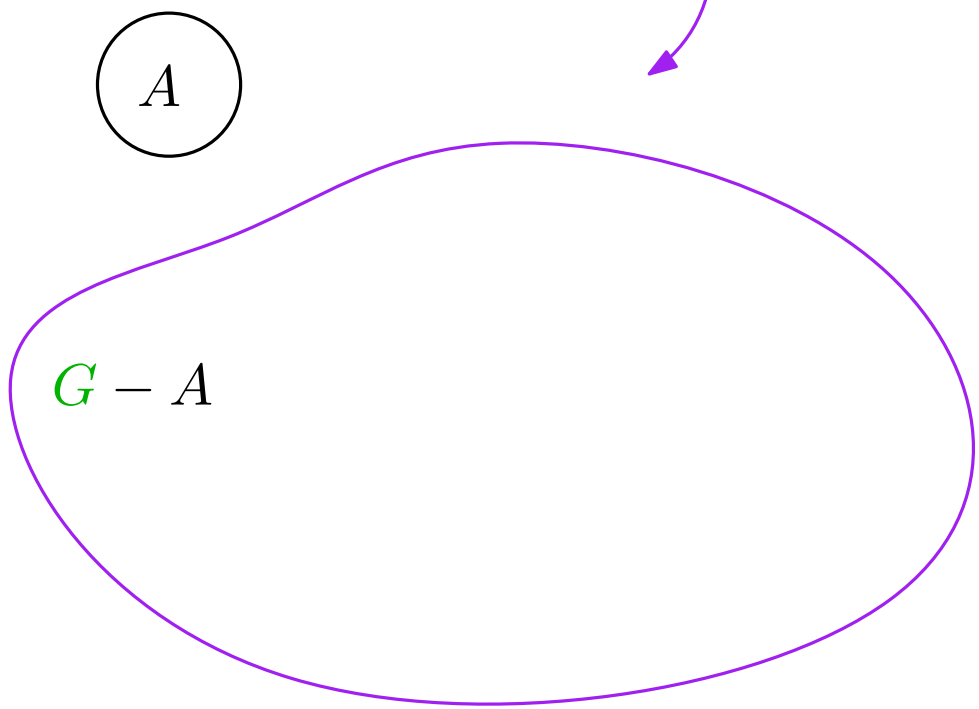

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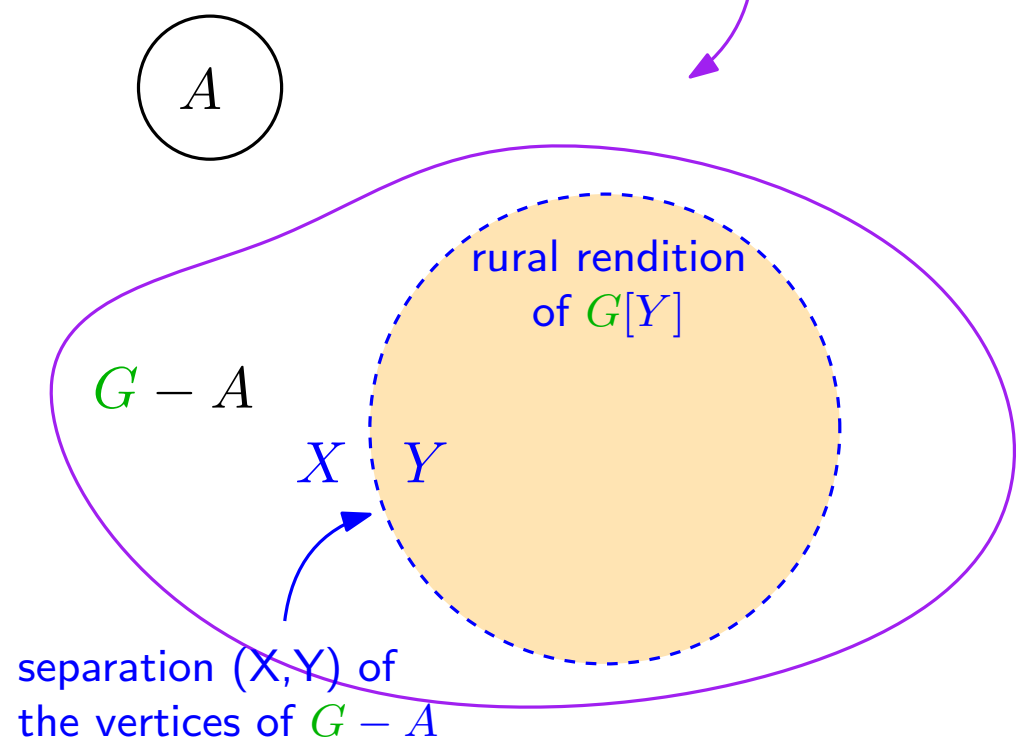

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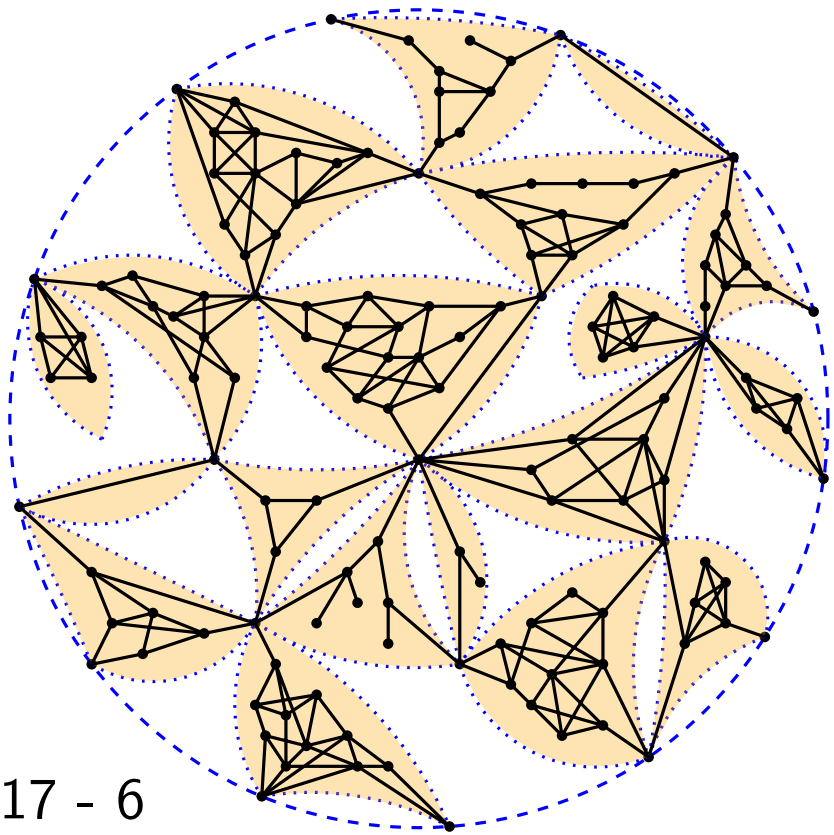

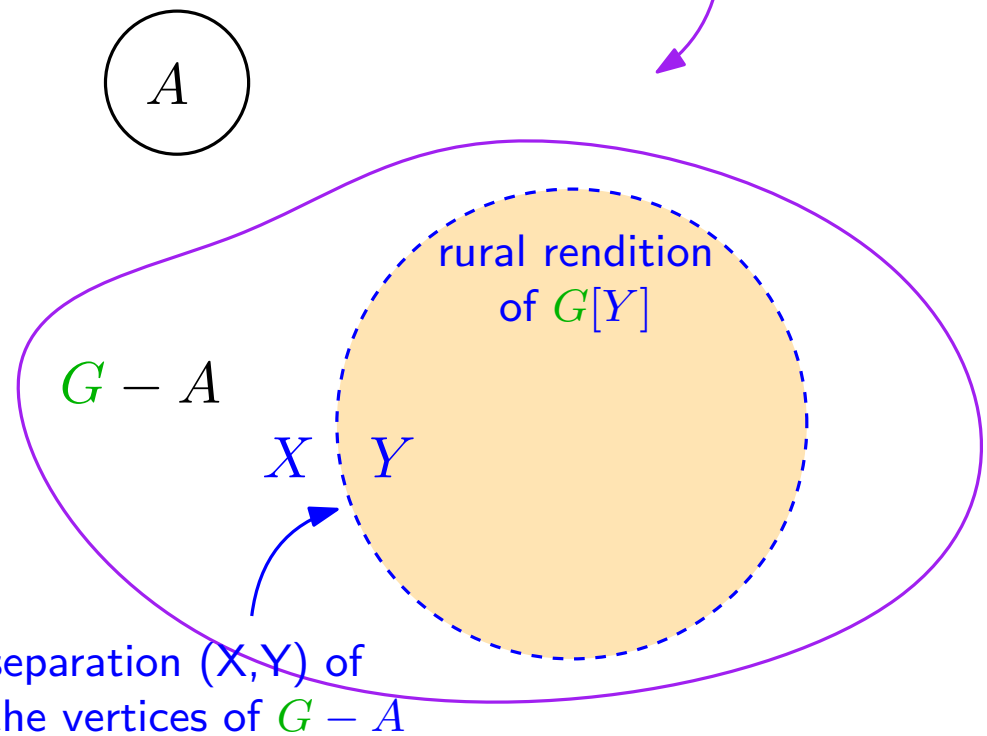

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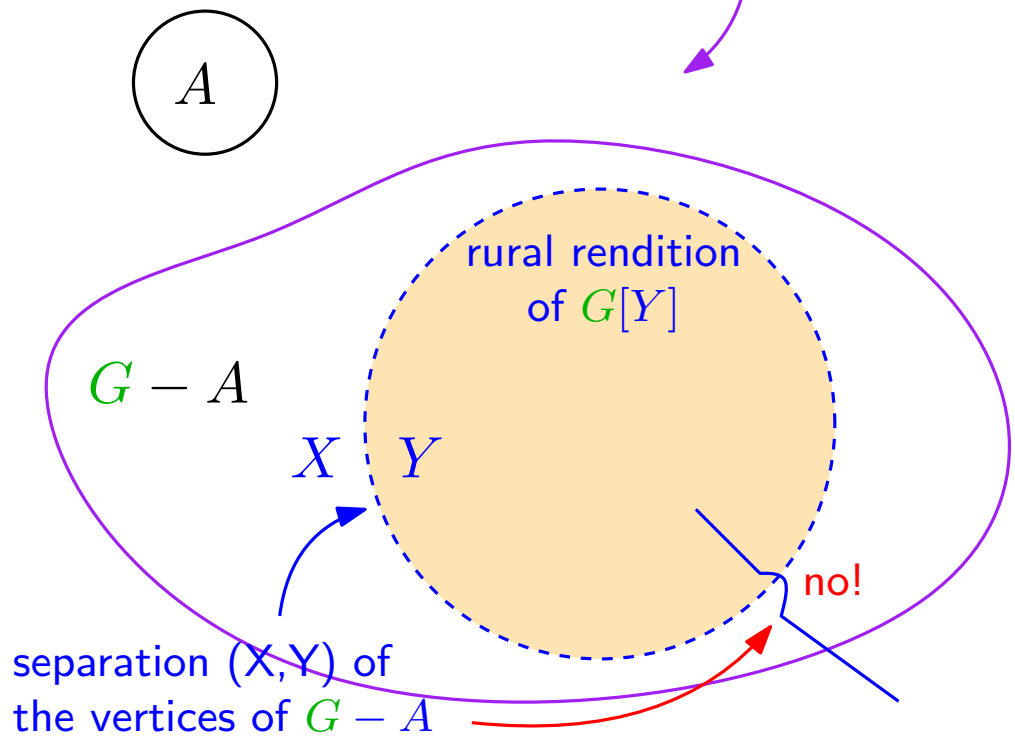

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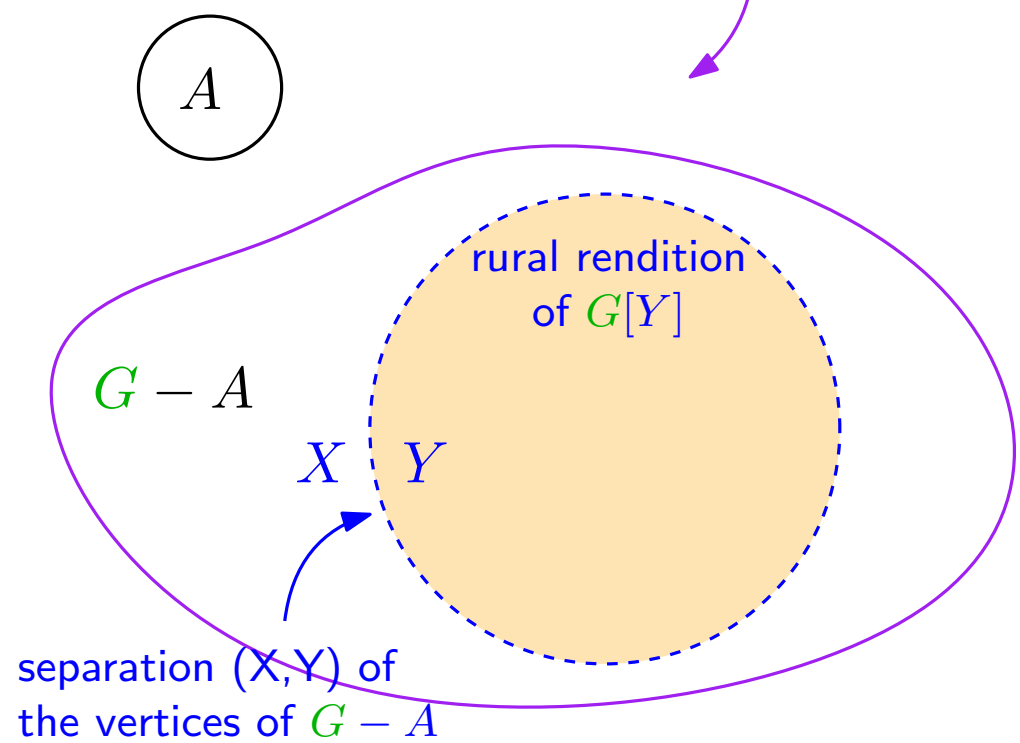

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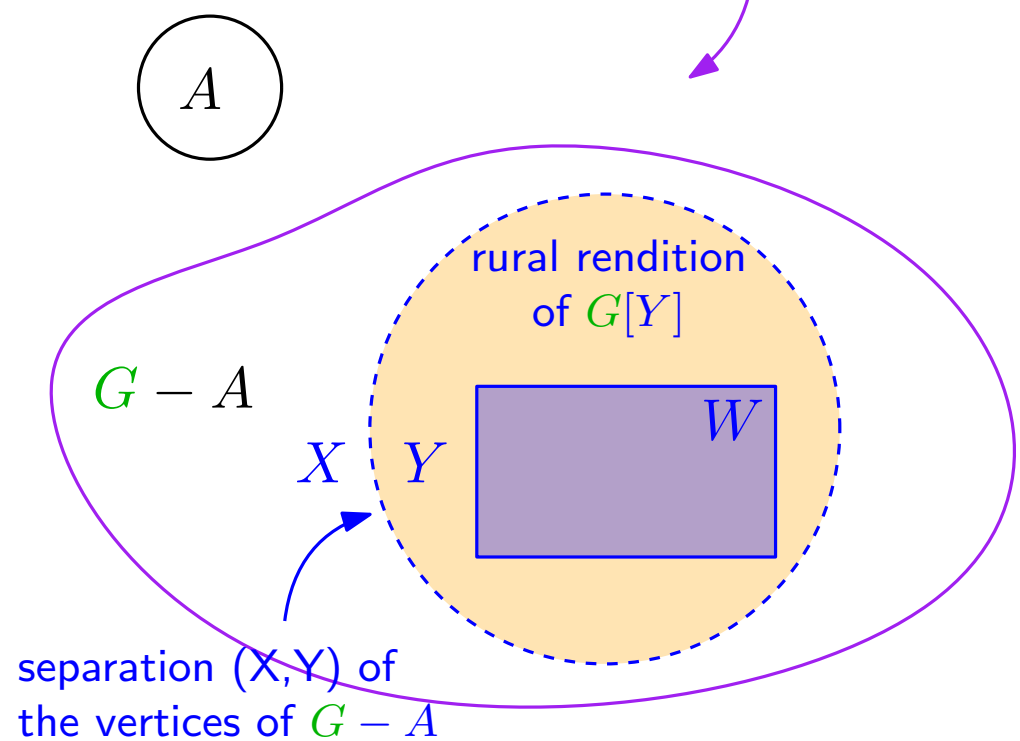

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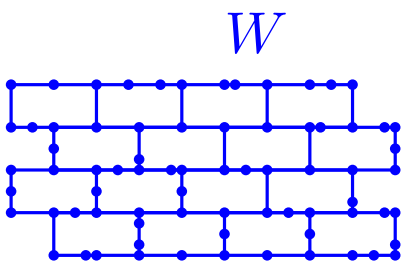

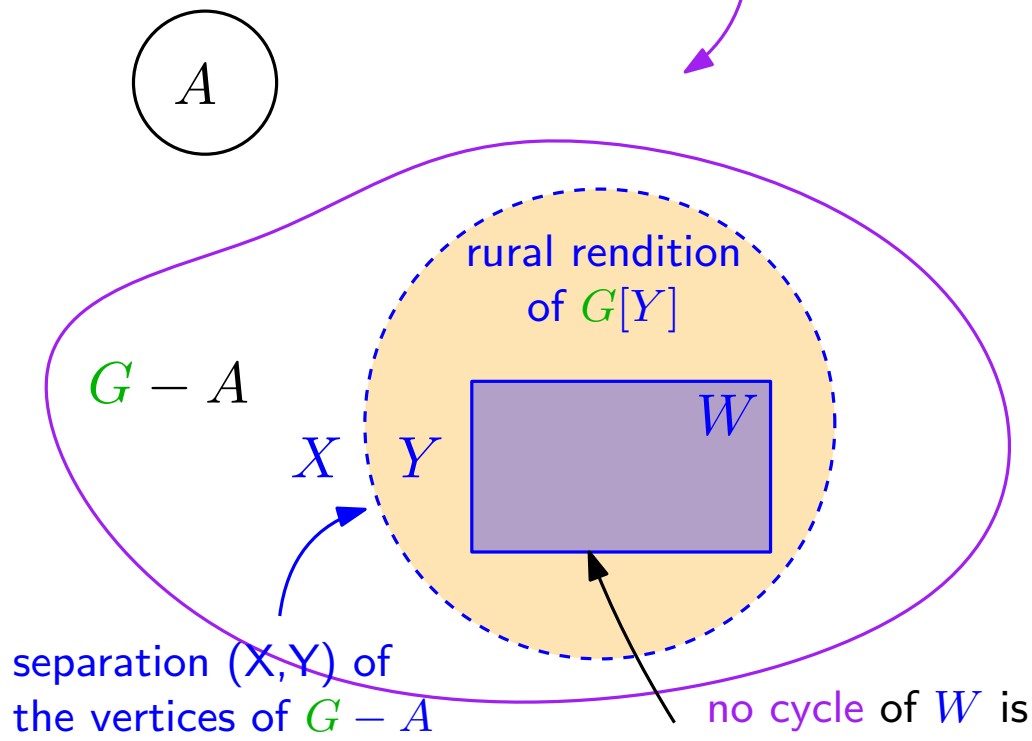

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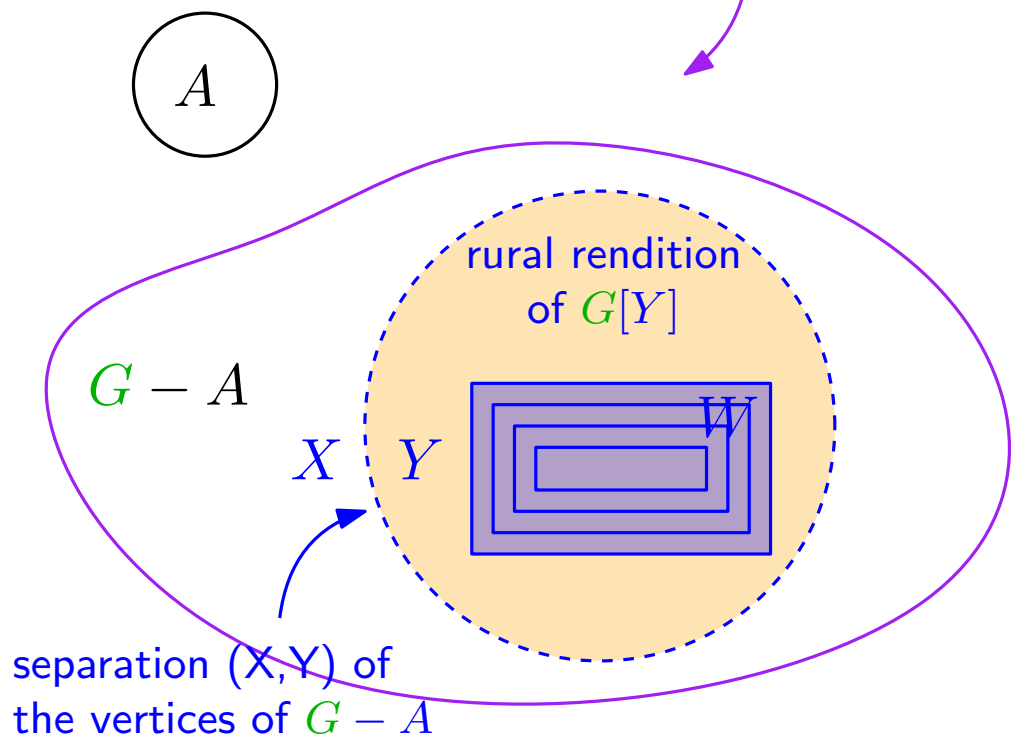

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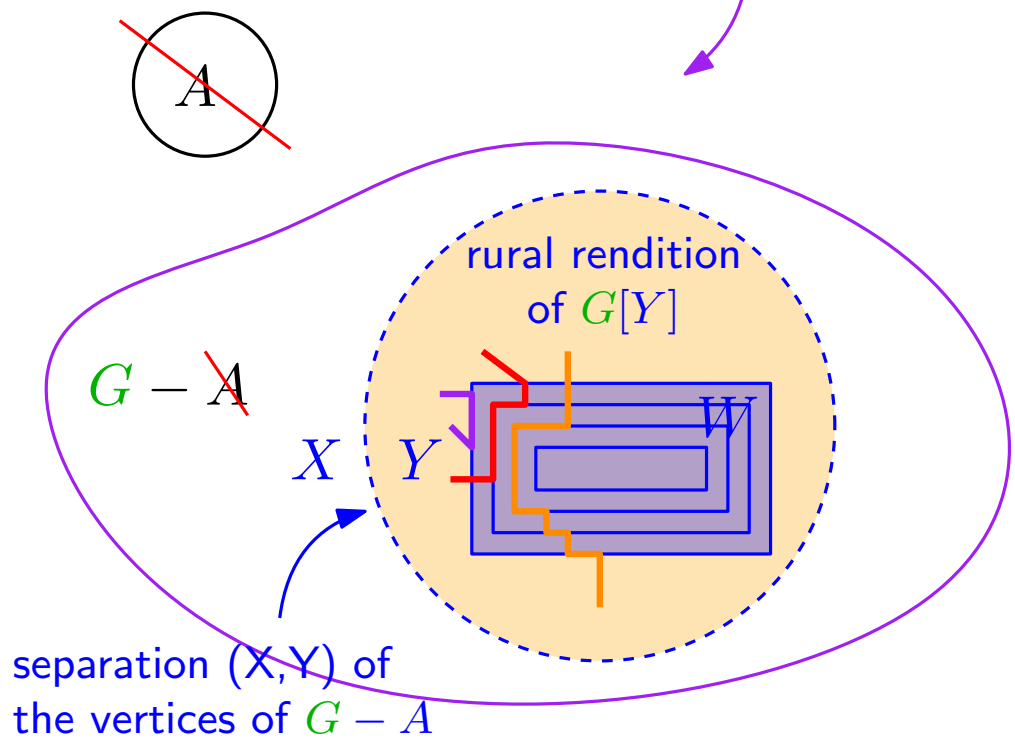

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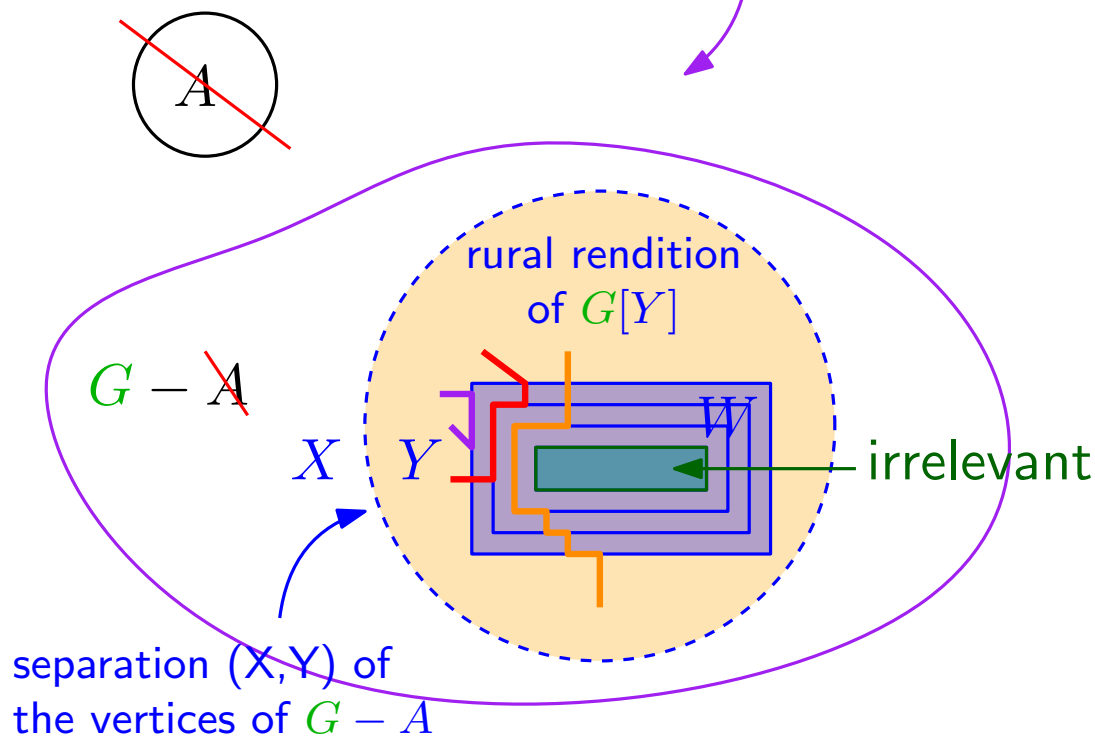

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Flat Wall theorem

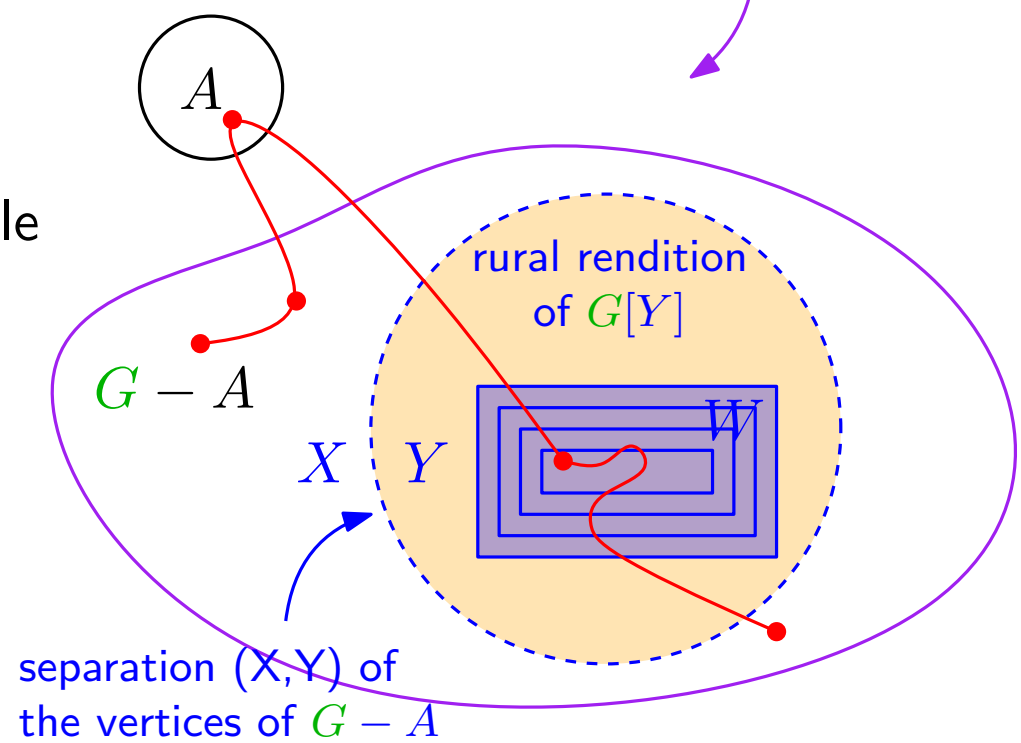
FPT algo in k and t

A graph G either has:

- treewidth at most $f(k,t)$, or
- a clique of size t as a minor, or
- a vertex set $A \leq g(t)$ and a wall W of height k s.t. W is a **flat wall** of $G - A$.

Problem if $A \neq \emptyset$:

a path might go through A in W:
cannot always reroute through a cycle



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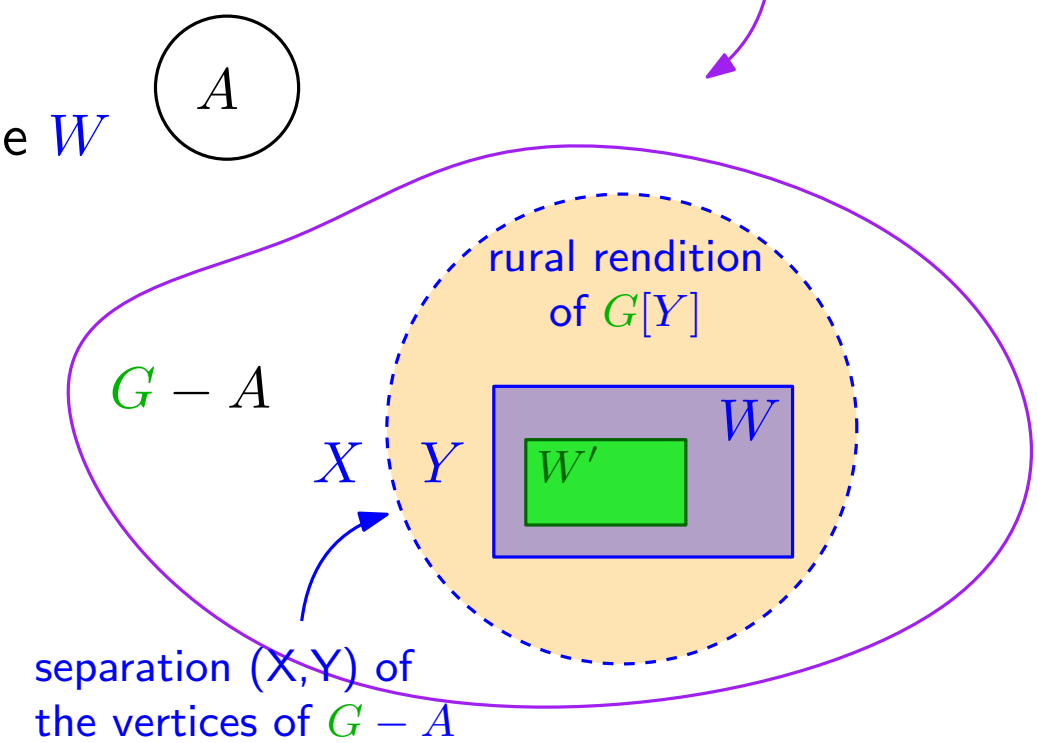

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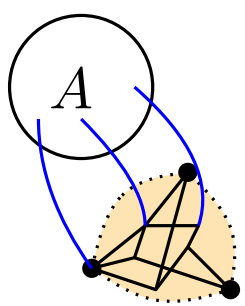
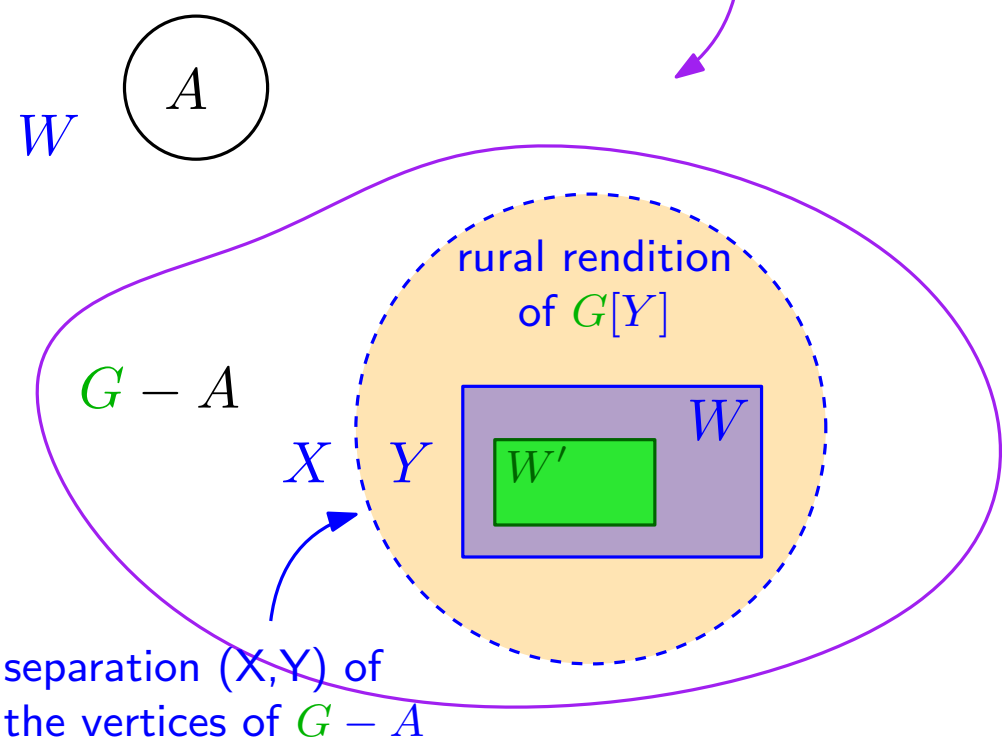
Folio of a cell $c \approx$ all minors present in $c + A$ of small size that are rooted at boundary$(c) + A$



folio $\approx$ all patterns in here



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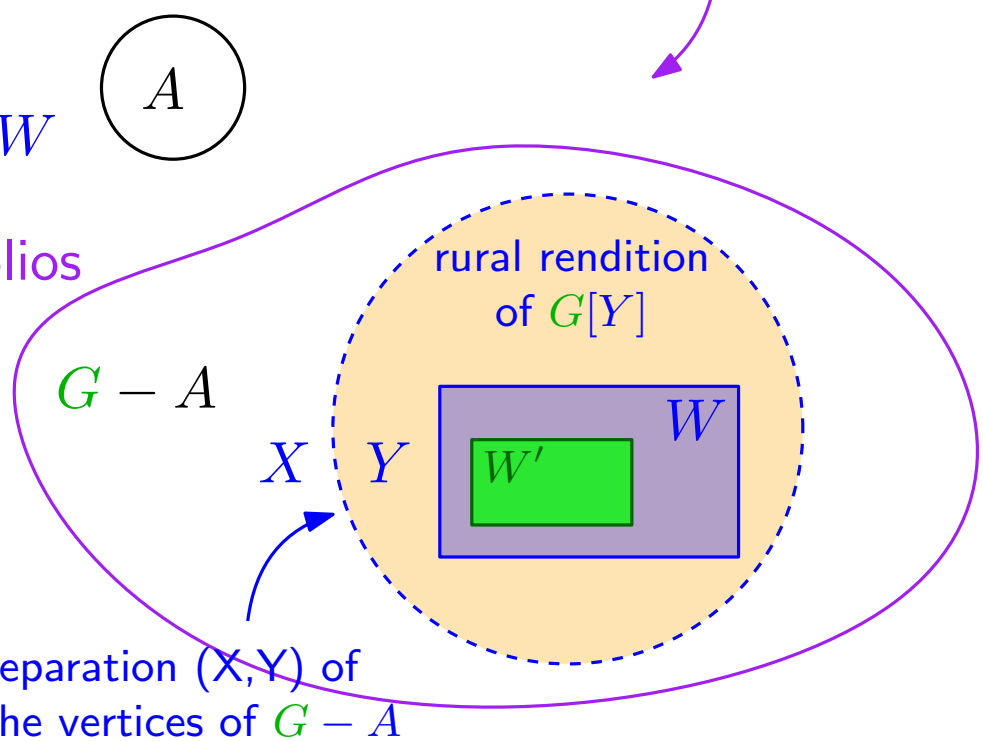

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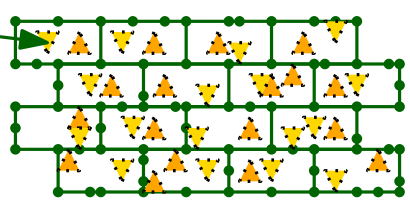

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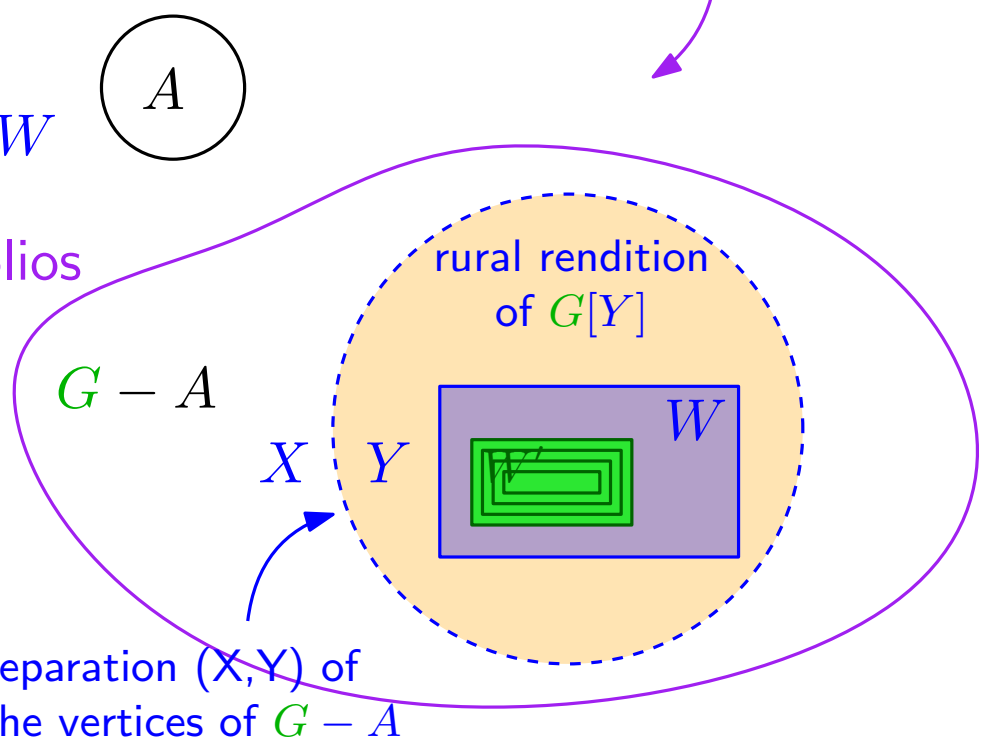

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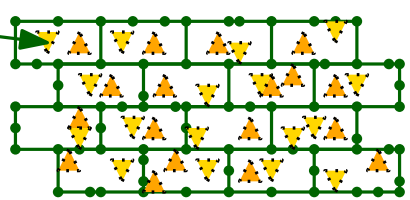

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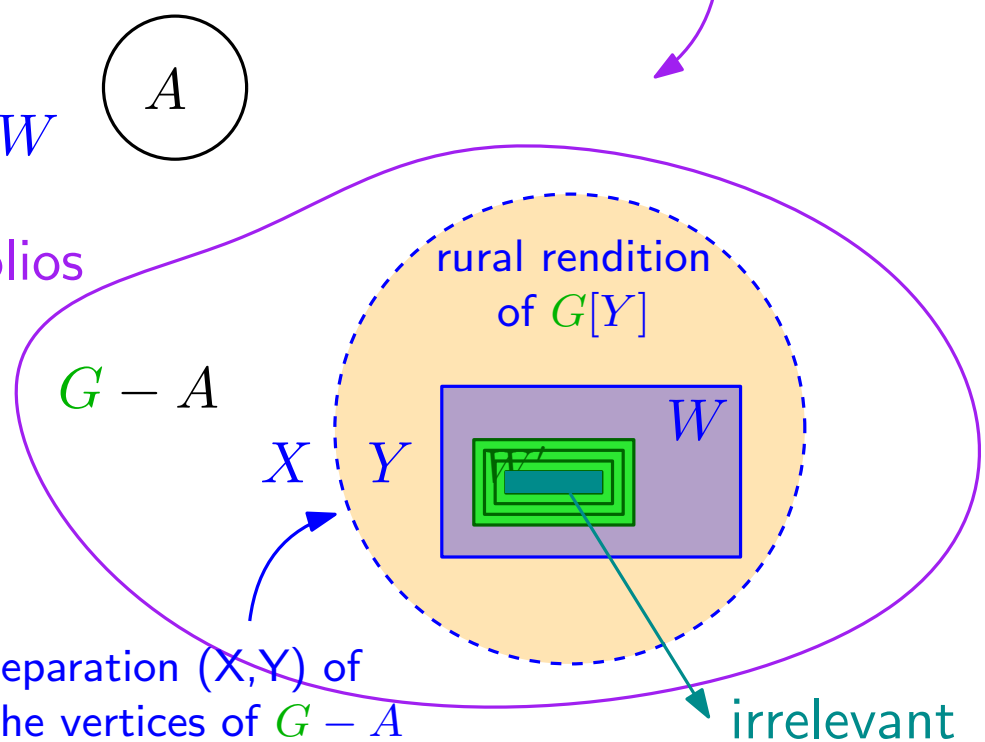

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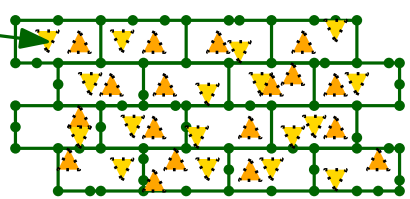

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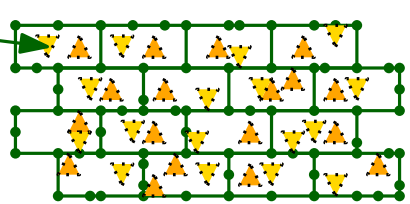
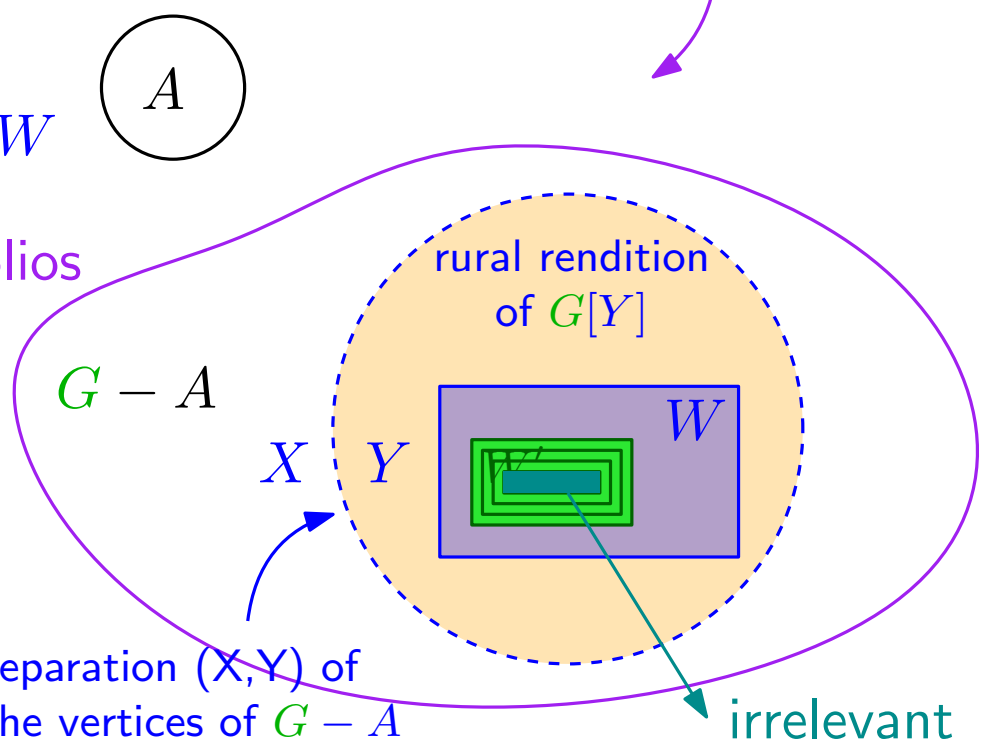
$\Longrightarrow$ can delete the central part of W' and recurse.

a brick





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Big open problem: $f(|V(H)|) \cdot n$?

Goal of the Graph Minor series:

Wagner's conjecture

[Graph Minors XX]

Wagner's conjecture [Wagner, 1937]

Graphs are **well quasi ordered** under the minor relation.

[Graph Minors XX] 2004

Graphs are **well quasi ordered** under the minor relation.

Wagner's conjecture [Wagner, 1937]

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Graphs are **well quasi ordered** under the minor relation.

What does it mean and why is it so important?

A **quasi order** $\preceq$ is a binary relation that is both reflexive ($x \preceq x$) and transitive ($x \preceq y$ and $y \preceq z \implies x \preceq z$)

$\implies$ Minor containment defines a quasi ordering on graphs:
if $G_1 \preceq_{\mathsf{m}} G_2$ and $G_2 \preceq_{\mathsf{m}} G_3$, then $G_1 \preceq_{\mathsf{m}} G_3$

An **antichain** is a set of pairwise non-comparable elements ($x \not\preceq y$ and $y \not\preceq x$).

A quasi order $\preceq$ defined on a set X is a **well quasi order** (**WQO**) if there is no infinite strictly decreasing sequence and no infinite antichain.

Are those WQO?

- $(\mathbb{N}, \leq)$
- $(\mathbb{R}, \leq)$
- $(\mathbb{N}^2, \preceq)$, where $(x, y) \preceq (x', y')$ iff ($x \leq x'$ and $y \leq y'$)
- graphs under the induced subgraph relation (delete vertices)
- graphs under the subgraph relation (delete vertices and edges)

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- $(\mathbb{N}, \leq)$ — yes
- $(\mathbb{R}, \leq)$ — no
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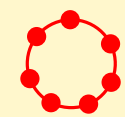

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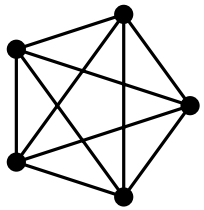
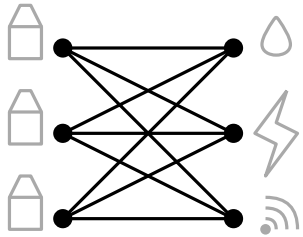
An **obstruction** of a minor-closed graph class $\mathcal{H}$ is a graph F that do not belong to $\mathcal{H}$, but whose minors all belong to $\mathcal{H}$.

$\Longrightarrow$ Obs(planar) $= \{K_5, K_{3,3}\}$





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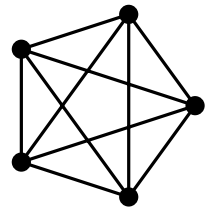

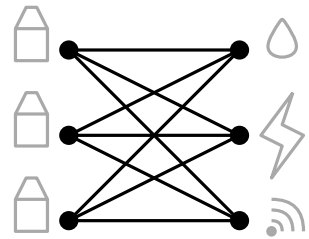

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One can check whether a graph H is a minor of G in time $f(|V(H)|) \cdot n^3$.

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Membership to $\mathcal{H}$

One can check whether a graph belongs to a minor-closed graph class $\mathcal{H}$ in time $\mathcal{O}_{\mathcal{H}}(n^3)$.

Proof

- List all obstructions of $\mathcal{H}$ (finite number ℓ): $F_1, \ldots, F_\ell$.
- for $1 \leq i \leq \ell$, check whether F_i is a minor of G.
- If there is i s.t. F_i is a minor of G, then $G \notin \mathcal{H}$.
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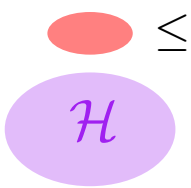

VERTEX DELETION TO $\mathcal{H}$

Input: A graph G, $k \in \mathbb{N}$.

Question: Is there a set S of size $\leq k$ s.t. $G - S \in \mathcal{H}$?

- $\mathcal{H}$= edgeless $\rightarrow$ VERTEX COVER
- $\mathcal{H}$= forest $\rightarrow$ FEEDBACK VERTEX SET
- $\mathcal{H}$= planar $\rightarrow$ PLANARIZATION

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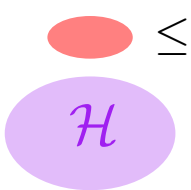

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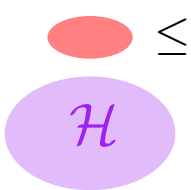

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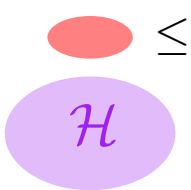

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$\Longrightarrow$ VERTEX DELETION TO $\mathcal{H}$ is solvable in time $f(k) \cdot n^{1+o(1)}$.

Conclusion

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... **but**

- requires lots of definitions that are not so easy to understand nor to use
- From Wikipedia, the free encyclopedia

  A **galactic algorithm** is an algorithm with record-breaking theoretical (asymptotic) performance, but which is not used due to practical constraints. Typical reasons are that the performance gains only appear for problems that are so large they never occur, or the algorithm's complexity outweighs a relatively small gain in real-world performance. Galactic algorithms were so named by Richard Lipton and Ken Regan,[1] because they will never be used on any data sets on Earth.

  Sub-graphs [edit]

  The problem of deciding whether a graph G contains H as a minor is NP-complete in general, but where H is fixed, it can be solved in polynomial time. The running time for testing whether H is a minor of G in this case is $O(n^2)$,[9] where n is the number of vertices in G and the big O notation hides a constant that depends superexponentially on H. The constant is greater than $2 \uparrow\uparrow (2 \uparrow\uparrow (2 \uparrow\uparrow (h/2)))$ in Knuth's up-arrow notation, where h is the number of vertices in H.[10] Even the case of $h = 4$ cannot be reasonably computed as the constant is greater than 2 pentated by 4, or 2 tetrated by 65536, that is, $2 \uparrow\uparrow\uparrow 4 = {}^{65536}2 = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{65536}$.

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Questions?

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